Information and Computation 208 (2010) 1398–1420

Contents lists available at ScienceDirect

Information and Computation

journal homepage: www.elsevier.com/locate/ic

Complete sets of cooperations

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ARTICLE INFO

Article history:
Available online 15 May 2010

Keywords:
Coalgebra
Coinduction
Infinite data structures
Hidden algebra
Cooperations
Observational/simple/minimal coalgebra

ABSTRACT

The structure map turning a set into the carrier of a final coalgebra is not unique. This fact is well known, but commonly elided. In this paper, we argue that any such concrete representation of a set as a final coalgebra is potentially interesting on its own. We discuss several examples, in particular, we consider different coalgebra structures that turn the set of infinite streams into the carrier of a final coalgebra. After that we focus on coalgebra structures that are made up using so-called cooperations. We say that a collection of cooperations is complete for a given set X if it gives rise to a coalgebra structure that turns X into the carrier set of a subcoalgebra of a final coalgebra. Any complete set of cooperations yields a coalgebraic proof and definition principle. We exploit this fact and devise a general definition scheme for constants and functions on a set X that is parametrical in the choice of the complete set of cooperations for X.

1. Introduction

It is well known that coalgebras provide a framework for studying infinite data structures, such as streams and trees, in a uniform way. The theory of coalgebras is formulated in category theoretic terms. Therefore coalgebras are usually studied "up-to-isomorphism", e.g., one talks about *the* final coalgebra of a functor because it is determined uniquely up-to-isomorphism. When reasoning about a concrete type of coalgebras one then has a certain "canonical" representation of the final coalgebra in mind. For the stream functor $A \times \mathrm{Id}$ the final coalgebra is usually given by the set of infinite A-streams A^ω together with the usual operations head and tail. There are, however, infinitely many ways of turning A^ω into the final stream coalgebra – we will discuss some of them in the paper. The point we want to make is that each of these representations of the final coalgebra is potentially interesting on its own as each of them yields a different proof and definition principle.

More generally, we consider not only the various representations of a given set X as a final coalgebra of some kind, but also its representations as a subcoalgebra of a final coalgebra. We call such a representation of a set X *observational* for X. Any subcoalgebra of a final coalgebra has two crucial properties: strong extensionality and what we call relative finality. The first property is the basis for a proof principle for observational coalgebra structures and the second one is the key for the coinductive definition of constants and functions for observational coalgebra structures.

In this paper, we first introduce the notion of an observational coalgebra and then motivate it with various examples. After that, in Section 4, we provide a simpler, syntactic version of the notion of an observational coalgebra by using the terminology from [5] of a *cosignature* and of a *cooperation*. We call a collection of cooperations *complete* for some set if it turns this set into an observational coalgebra. After having defined these notions we turn to the discussion of the proof principle and of the definition scheme.

In Section 5, we discuss the proof principle for a complete set of cooperations and demonstrate with an example that a clever choice of cooperations for the set of streams can simplify proofs. After that, in Section 6 we develop a definition scheme

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1 Supported by NWO under FOCUS/BRICKS Grant 642.000.502.

0890-5401/$ - see front matter © 2010 Published by Elsevier Inc.
doi:10.1016/j.ic.2009.10.009

for constants and functions on a given set that is equipped with a complete set of cooperations. The main advantage of this scheme lies in the fact that it works for various types of objects as we demonstrate at the end of Section 6. In particular, our scheme can be applied to sets of objects that have no "nice", purely coalgebraic representation, such as bi-infinite streams. We conclude our paper in Section 7 by linking our research to related work, in particular, to the field of hidden algebra, and by the discussion of future work.

The paper is an extended version of [12]. The main changes are the addition of more examples and a generalisation of the definition scheme from so-called basic cosignatures to arbitrary ones.

1.1. Related work

There is a close connection between our work on the one hand and existing work in hidden algebra and observational specification on the other hand. A strong link between coalgebra and hidden algebra has been established in a series of articles by Cîrstea [4,5] in which it is shown that, under the assumption that any operation of a hidden algebra signature has at most one argument of hidden sort, hidden algebras can be seen as coalgebras. Our notion of a complete set of cooperations is inspired by the notion of a cobasis in hidden algebra (see, e.g., [8,14,15]) and our definition scheme in Section 6 has some similarity with the one in [3]. Throughout the paper we will tell the reader precisely which of our notions and results generalise/relate to similar ones in hidden algebra.

At the same time, there are also many differences between our approach and that of hidden algebra, and we believe that our results have importance on their own, for the following reasons.

1.2. Coalgebraic implications

Even if not completely new in hidden algebra, the concept of a complete set of cooperations is a novel contribution to the theory of coalgebras. We link the technically rather involved notion of a cobasis to the basic and conceptually clear notion of a so-called "observational coalgebra".

1.3. Different perspective

Most of the work within hidden algebra and in observational specification focuses on proofs. While we also mention in our paper the proof principle for complete sets of cooperations, we pay much more attention to the arising definition scheme.

1.4. Different definition schemes

This definition scheme is to the best of our knowledge new in its generality: the afore mentioned scheme that has been proposed in [3] does not allow for the *simultaneous* definition of functions. The price we have to pay for this gain in generality is that we have to be more restrictive concerning the structure of the contexts that are allowed on the right hand side of a defining equation. In particular, we do not make use of any kind of context induction (see, e.g., [11]). The latter is difficult to use in our setting as a simultaneous definition of several functions can easily lead to unwanted circularities.

1.5. Different notions

Finally one should note that our notion of a complete set of cooperations is similar to the one of a cobasis but the two notions do not coincide. One way of formulating the difference is by saying that cobases are defined *syntactically* with respect to a certain specification (w.r.t. a certain "behavioural theory"). Complete sets of cooperations on the other hand are defined *semantically*. We will return to this issue in the conclusions of our paper.

In summary, there are both close connections and differences between the theory of coalgebra and hidden algebra, and we believe it is beneficial for both communities to learn from each others perspective.

2. Preliminaries

We assume that the reader is familiar with the basic notions from category theory and universal coalgebra. The purpose of the following basic definitions is mainly to fix our notation.

Definition 1. We define the range of a function $f : X \to Y$ by putting $\mathrm{range}(f) := \{y \in Y \mid \exists x \in X. f(x) = y\}$. Given a set $\{\nu_1, \dots, \nu_n\}$ we write $\overline{\nu}$ as an abbreviation of $\nu_1, \dots, \nu_n$.

In this paper, we consider coalgebras for functors on the category **Set** of sets and functions. Coalgebras for such a functor $G : \mathbf{Set} \to \mathbf{Set}$ consist of a set X together with a function $\gamma : X \to GX$.

Definition 2. Let G : **Set** $\to$ **Set** be a functor. A set X together with a function $\gamma : X \to GX$ is a G-*coalgebra*. A function $f : X_1 \to X_2$ is a G-*coalgebra morphism* from $\mathbb{X}_1 = (X_1, \gamma_1 : X \to GX)$ to $\mathbb{X}_2 = (X_2, \gamma_2 : X \to GX)$ if $\gamma_2 \circ f = Gf \circ \gamma_1$. In case the final G-coalgebra exists we denote by $\varphi^{\mathbb{X}_1}$ the unique coalgebra morphism from (X_1, γ_1) into the final G-coalgebra.

Intuitively speaking, G-coalgebras can be seen as state-based systems that generate possibly infinite behaviour. For example, if $G_{\mathbb{R}}$ is the functor mapping a set X to the set $\mathbb{R} \times X$, the behaviour of a state $x \in X$ can be easily seen to be an infinite $\mathbb{R}$-stream $\varphi(x) \in \mathbb{R}^\omega$: For $x \in X$ with $\gamma(x) = \langle r, x' \rangle$ we obtain $\varphi(x)$ by taking r to be the first element of $\varphi(x)$ followed by the $\mathbb{R}$-stream $\varphi(x')$ that corresponds to the behaviour of the successor x' of x. For a detailed introduction to coalgebra as a theory of state-based systems the reader is referred to [16]. Given a G-coalgebra (X, γ) we can prove that two states $x_1, x_2 \in X$ have the same behaviour by showing that they are bisimilar in the following sense.

Definition 3. A relation $R \subseteq X_1 \times X_2$ is a G-*bisimulation* between (X_1, γ_1) and (X_2, γ_2) if there is a map $\mu : R \to GR$ such that the projection maps $\pi_i : R \to X_i$ are G-coalgebra morphisms $\pi_i : (R, \mu) \to (X_i, \gamma_i)$ for $i = 1, 2$. For G-coalgebra states $x_1 \in X_1$ and $x_2 \in X_2$ we say x_1 and x_2 are G-bisimilar (Notation: $x_1 \leftrightarroweq_G x_2$) if there is a G-bisimulation $R \subseteq X_1 \times X_2$ such that $(x_1, x_2) \in R$.

In the above $\mathbb{R}$-stream example a relation $R \subseteq X \times X$ on a $G_{\mathbb{R}}$-coalgebra (X, γ) is a $G_{\mathbb{R}}$-bisimulation iff for any two states x_1, x_2 we have $(x_1, x_2) \in R$ implies $r_1 = r_2$ and $(x_1', x_2') \in R$ where $\gamma(x_1) = \langle r_1, x_1' \rangle$ and $\gamma(x_2) = \langle r_2, x_2' \rangle$.

In our paper, we only consider set functors G for which a final G-coalgebra exists.

Definition 4. Let G : **Set** $\to$ **Set** be a functor. A G-coalgebra (Ω_G, ω_G) is called *final* if for all G-coalgebras $\mathbb{X} = (X, \gamma)$ there exists a unique coalgebra morphism $\varphi^{\mathbb{X}} : X \to \Omega_G$.

As mentioned above, final coalgebras are unique up-to-isomorphism. The final G-coalgebra can be seen as a system that contains for any possible behaviour of a G-coalgebra exactly one state. In the $\mathbb{R}$-stream example, the final $G_{\mathbb{R}}$-coalgebra has the set $\mathbb{R}^\omega$ as set of states and the function $\omega : \mathbb{R}^\omega \to \mathbb{R} \times \mathbb{R}^\omega$ is given by $\omega(r_0 r_1 r_2 r_3 \ldots) = \langle r_0, r_1 r_2 r_3 \ldots \rangle$. Given any other $G_{\mathbb{R}}$-coalgebra (X, γ) it is easy to see that the function $\varphi : X \to \mathbb{R}^\omega$ that maps a state x to its behaviour $\varphi(x)$ as described above is the unique coalgebra morphism into the final $G_{\mathbb{R}}$-coalgebra.

We refer the reader to [16] for a description of the final coalgebras of large families of functors, including all those that will occur in the present paper.

3. Observational coalgebra structures

In this section we introduce the notion of an observational coalgebra structure. Despite the fact that this is a rather simple notion we hope to demonstrate throughout the remainder of the paper its usefulness.

Definition 5. Let X be a set and let G : **Set** $\to$ **Set** be a functor for which the final G-coalgebra (Ω_G, ω_G) exists. We call $\gamma : X \to GX$ *observational* for X if the unique morphism $\varphi : X \to \Omega_G$ into the final G-coalgebra is injective. In this case the coalgebra (X, γ) will be called *observational*.

Remark 6. The concept of an observational coalgebra is nothing essentially new. Observational coalgebras are merely (isomorphic to) subcoalgebras of some final coalgebra and, under the condition that the final coalgebra for the functor G : **Set** $\to$ **Set** exists, observational G-coalgebras are exactly the simple G-coalgebras from [16] or the minimal G-coalgebras from [10]. The novelty of our work lies in the fact, that we focus on the various *observational coalgebra structures* that turn a given set into an observational coalgebra.

In order to motivate this definition we provide a number of examples.

Example 7

(1) Let (X, ω) be the final G-coalgebra for a functor G : **Set** $\to$ **Set**. Then ω is observational for X.
(2) Consider the set $\mathbb{N}$ of natural numbers and let $P : \mathbb{N} \to 1 + \mathbb{N}$ be the predecessor map, i.e., $P(n + 1) := n$ and $P(0) := * \in 1$. Then P is observational for $\mathbb{N}$: P turns $\mathbb{N}$ into a coalgebra for the functor $1 + \mathrm{Id}$ and this functor has as final coalgebra the set $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ together with the "extended" predecessor map $\bar{P}$, where $\bar{P}(n) := P(n)$ for all $n \in \mathbb{N}$ and $\bar{P}(\infty) := \infty$. The obvious embedding of $\mathbb{N}$ into $\bar{\mathbb{N}}$ is the injective coalgebra morphism from $(\mathbb{N}, P)$ into the final $1 + \mathrm{Id}$-coalgebra $(\bar{\mathbb{N}}, \bar{P})$.
(3) Let $p > 0$ be a natural number and let

$$P_p : \mathbb{N} \longrightarrow 1 + \{0, \ldots, p - 1\} \times \mathbb{N}$$

be the map defined by

$$P_p(n) := * \qquad \text{if} \quad n = 0$$
$$P_p(n) := (n \bmod p, \lfloor \tfrac{n}{p} \rfloor) \qquad \text{if} \quad n > 0,$$

where $\lfloor - \rfloor$ denotes the "floor function" or "entier function" that maps a rational number q to the greatest integer below q, i.e., $\lfloor q \rfloor := \max\{z \in \mathbb{Z} \mid z \le q\}$. Then P_p is observational for $\mathbb{N}$ for all $p > 0$. The carrier of the final coalgebra of $G = 1 + \{0, \ldots, p-1\} \times \mathrm{Id}$ is the set $p^\infty = p^* \cup p^\omega$ where $p = \{0, \ldots, p-1\}$. The final map $\varphi : \mathbb{N} \to p^\infty$ maps a natural number n to its p-adic representation starting with the least significant digit. Therefore φ is obviously injective.

(4) Let A be a set and $A^{\mathbb{Z}}$ be the set of bi-infinite streams over A. Then the map $\langle h, l, r \rangle : A^{\mathbb{Z}} \to A \times A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ is observational for $A^{\mathbb{Z}}$. Here $\langle h, l, r \rangle$ is the function that maps a given bi-infinite stream $\tau = \ldots a_{-3}a_{-2}a_{-1}\underline{a}_0a_1a_2a_3\ldots$ to its head $h(\tau) = a_0$, its left neighbour $l(\tau) = \ldots a_{-4}a_{-3}a_{-2}\underline{a}_{-1}a_0a_1a_2\ldots$ and its right neighbour $r(\tau) = \ldots a_{-2}a_{-1}a_0\underline{a}_1a_2a_3a_4\ldots$.

(5) Consider the functor $G(X) = \mathbb{R} \times X$. The final coalgebra of G consists of the set $\mathbb{R}^\omega$ of real-valued streams together with the familiar coalgebra map $< h, t >: \mathbb{R}^\omega \to \mathbb{R} \times \mathbb{R}^\omega$ of head and tail:

$$h(\sigma) = \sigma(0), \qquad t(\sigma) = (\sigma(1), \sigma(2), \sigma(3), \ldots)$$

Obviously $< h, t >$ is observational for $\mathbb{R}^\omega$.

(6) We can also supply $\mathbb{R}^\omega$ with an alternative coalgebra structure as follows. For $\sigma \in \mathbb{R}^\omega$ we define

$$\Delta\sigma = (\sigma(1) - \sigma(0), \sigma(2) - \sigma(1), \sigma(3) - \sigma(2), \ldots)$$

(cf., [13,17]). We claim that the coalgebra map

$$< h, \Delta >: \mathbb{R}^\omega \to \mathbb{R} \times \mathbb{R}^\omega \qquad \sigma \mapsto < \sigma(0), \Delta\sigma >$$

is observational for $\mathbb{R}^\omega$. The unique morphism

$$\varphi : (\mathbb{R}^\omega, < h, \Delta >) \to (\mathbb{R}^\omega, < h, t >)$$

is given by

$$\varphi(\sigma) = ((\Delta^{(0)}\sigma)(0), (\Delta^{(1)}\sigma)(0), (\Delta^{(2)}\sigma)(0), \ldots)$$

where $\Delta^{(0)}\sigma = \sigma$ and $\Delta^{(n+1)}\sigma = \Delta(\Delta^{(n)}\sigma)$. One can easily verify that φ is injective.

(7) Here is yet another coalgebra structure on $\mathbb{R}^\omega$. For $\sigma \in \mathbb{R}^\omega$, we define

$$\frac{d\sigma}{dX} = (\sigma(1), 2 \cdot \sigma(2), 3 \cdot \sigma(3), \ldots)$$

The coalgebra map $< h, d/dX >: \mathbb{R}^\omega \to \mathbb{R} \times \mathbb{R}^\omega$ is observational for $\mathbb{R}^\omega$ as the unique morphism

$$\varphi : (\mathbb{R}^\omega, < h, d/dX >) \to (\mathbb{R}^\omega, < h, t >)$$

which is given by

$$\varphi(\sigma) = (\sigma(0), \sigma(1), 2! \cdot \sigma(2), 3! \cdot \sigma(3), \ldots)$$

is injective.

(8) Let $F = G_\Sigma$ for a finite cosignature Σ (cf., Definition 16 below) and let (A, α) be the initial F-algebra. Then $\alpha^{-1} : A \to FA$ is observational for A. The claim is a consequence of a more general result in [2]. Note that this example generalizes (2) above.

Two properties of observational coalgebras will play a central rôle in our paper: given an observational coalgebra structure $\gamma : X \to GX$ for some set X, we have that the G-coalgebra (X, γ) is strongly extensional (Proposition 10) and relatively final (Proposition 12). The first property gives rise to a proof principle for elements of observational coalgebras, the second property is the basis of the definition scheme which we develop in Section 6.

Remark 8. Propositions 10 and 12 below are in fact easy consequences of the fact that an observational coalgebra structure γ represents a subcoalgebra of a final coalgebra. We provided the easy proofs in order to keep our paper as self-contained as possible.

Definition 9. Let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor and let $\mathbb{X} = (X, \gamma)$ be a G-coalgebra. We say $\mathbb{X}$ is *strongly extensional* iff for all $x_1, x_2 \in X$ we have $x_1 \leftrightarrows x_2$ iff $x_1 = x_2$.

Proposition 10. *Let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor with final coalgebra and let $\mathbb{X} = (X, \gamma)$ be a G-coalgebra. If γ is observational for X then $\mathbb{X}$ is strongly extensional.*

Proof. Let $\mathbb{X} = (X, \gamma)$ be observational and consider a G-bisimulation $R \subseteq X \times X$ with projections π_1 and π_2. Furthermore let φ be the unique morphism from $\mathbb{X}$ into the final coalgebra. By finality, we have $\varphi \circ \pi_1 = \varphi \circ \pi_2$. Because $\mathbb{X}$ is observational, φ is injective, whence $\pi_1 = \pi_2$. As a consequence, any two G-bisimilar elements of X are equal. For the converse, note that the identity relation is a G-bisimulation. $\square$

Definition 11. Let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor with final coalgebra. A G-coalgebra $\mathbb{X} = (X, \gamma)$ is called *relatively final* if for all G-coalgebras $\mathbb{Y} = (Y, \delta)$ such that $\mathrm{range}(\varphi^{\mathbb{Y}}) \subseteq \mathrm{range}(\varphi^{\mathbb{X}})$ there is a unique G-coalgebra morphism $\iota : \mathbb{Y} \to \mathbb{X}$ with

$$\begin{tikzcd} X \arrow[r, "\varphi^{\mathbb{X}}"] & \Omega_G \\ Y \arrow[u, dashed, "\iota"] \arrow[ur, "\varphi^{\mathbb{Y}}"'] & \end{tikzcd}$$

Proposition 12. *Let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor with final coalgebra and let $\mathbb{X} = (X, \gamma)$ be a G-coalgebra. If γ is observational for X, then $\mathbb{X}$ is relatively final.*

Proof. Let $\mathbb{X}$ be an observational G-coalgebra, let $\mathbb{Y} = (Y, \delta)$ be a G-coalgebra and let $\varphi^{\mathbb{X}}$ and $\varphi^{\mathbb{Y}}$ be the coalgebra morphisms from $\mathbb{X}$ and $\mathbb{Y}$ into the final G-coalgebra. Furthermore we assume that $\mathrm{range}(\varphi^{\mathbb{Y}}) \subseteq \mathrm{range}(\varphi^{\mathbb{X}})$. We want to show that there is a unique G-coalgebra morphism ι from $\mathbb{Y}$ to $\mathbb{X}$. In order to show the existence of ι we define a function $\iota : Y \to X$ by putting for all $y \in Y$

$$\iota(y) := x \qquad \text{if } \varphi^{\mathbb{Y}}(y) = \varphi^{\mathbb{X}}(x).$$

This function is well defined because of the injectivity of $\varphi^{\mathbb{X}}$ and the fact that the range of $\varphi^{\mathbb{Y}}$ is contained in the range of $\varphi^{\mathbb{Y}}$. Clearly we have $\varphi^{\mathbb{Y}} = \varphi^{\mathbb{X}} \circ \iota$ which implies that ι is a coalgebra morphism because $\varphi^{\mathbb{X}}$ is injective (cf., [16, Lemma 2.4]). Uniqueness of ι follows also from the injectivity of $\varphi^{\mathbb{X}}$: any $\iota' : \mathbb{Y} \to \mathbb{X}$ has the property that $\varphi^{\mathbb{X}} \circ \iota' = \varphi^{\mathbb{Y}} = \varphi^{\mathbb{X}} \circ \iota$ and thus $\iota = \iota'$. $\square$

4. Complete sets of cooperations

The notion of an observational coalgebra is in general too abstract to work with. In this section we define the more concrete notion of a complete set of cooperations. We first introduce the notion of a cosignature and of a cooperation and then state when a given set of cooperations is complete.

4.1. Cosignatures

The notion of a cosignature that we are using is essentially the one from [5] with the difference that we consider only one "hidden sort" that corresponds to the set of coalgebra states.

Definition 13. Let $\mathcal{S} := \{S_j\}_{j \in J}$ be a family of sets ("observable sorts"). A basic $\mathcal{S}$-arity α is an element of the set $\mathcal{S}^* \times (\mathcal{S} \cup \{\mathsf{H}\})$, i.e., any basic $\mathcal{S}$-arity α is either of the form $(S_1 \ldots S_n, S)$ or of the form $(S_1 \ldots S_n, \mathsf{H})$, where H should be thought of as the "hidden sort". The set $\mathrm{Arity}(\mathcal{S})$ of $\mathcal{S}$-arities is defined as

$$\mathrm{Arity}(\mathcal{S}) = \{\alpha_1 + \cdots + \alpha_m \mid m \in \mathbb{N}, \alpha_i \text{ is a basic } \mathcal{S}\text{-arity}\}.$$ [2]

An $\mathcal{S}$-sorted cosignature consists of a set Σ of "cooperation" symbols and a function $a : \Sigma \to \mathrm{Arity}(\mathcal{S})$ that assigns to each $\sigma \in \Sigma$ its arity $a(\sigma) = \alpha_1 + \cdots + \alpha_m$. We call Σ *basic* if it contains only cooperation symbols σ of basic arity.

[2] Here $\alpha_1 + \cdots + \alpha_m$ denotes the word $\alpha_1 \ldots \alpha_m$ - the $+$'s have no formal meaning and are only there in order to make the structure of a given $\mathcal{S}$-arity more clear.

Definition 14. Let $\mathcal{S}$ be a family of sorts, let Σ be an $\mathcal{S}$-sorted cosignature and let X be a set. For each arity $\alpha_1 + \cdots + \alpha_k \in \mathrm{Arity}(\mathcal{S})$ we inductively define a corresponding set X_α by putting

$$X_{(S_1 \cdots S_k, S)} := S^{S_1 \times \cdots \times S_k} \qquad X_{(S_1 \ldots S_k, \mathsf{H})} := X^{S_1 \times \cdots \times S_k} \qquad X_{\alpha_1 + \alpha_2} := X_{\alpha_1} + X_{\alpha_2}$$

A *cooperation* of arity $a(\sigma) \in \mathrm{Arity}(\mathcal{S})$ is a function $f : X \to X_{a(\sigma)}$.

Notation 15. Given an $\mathcal{S}$-arity $\alpha_1 + \cdots + \alpha_m$ we will skip the canonical injection maps if no confusion is possible and simply write $y \in Y_{\alpha_i}$ instead of $y \in \kappa_i[Y_{\alpha_i}]$ where $\kappa_i : Y_{\alpha_i} \to Y_{\alpha_1} + \cdots + Y_{\alpha_m}$ denotes the canonical inclusion map.

Definition 16. Let $\mathcal{S}$ be a family of sorts and let Σ be an $\mathcal{S}$-sorted cosignature. A Σ-coalgebra $(X, \langle f_\sigma : \sigma \in \Sigma \rangle)$ consists of a set X and a collection of functions $\{f_\sigma : X \to X_{a(\sigma)}\}_{\sigma \in \Sigma}$. In other words, a Σ-coalgebra is a coalgebra for the functor

$$\begin{aligned} G_\Sigma : \mathbf{Set} &\to \mathbf{Set} \\ X \quad &\mapsto \quad \prod_{\sigma \in \Sigma} X_{a(\sigma)} \\ X \xrightarrow{h} Y \quad &\mapsto \quad \langle h_\sigma : \sigma \in \Sigma \rangle \end{aligned}$$

where $h_\sigma : X_{a(\sigma)} \to Y_{a(\sigma)}$ is defined in the obvious way. We call $g : Y_1 \to Y_2$ a Σ-coalgebra morphism from $(Y_1, \langle o^1_\sigma : \sigma \in \Sigma \rangle)$ to $(Y_2, \langle o^2_\sigma : \sigma \in \Sigma \rangle)$ if g is a G_Σ-coalgebra morphism.

Readers that are familiar with hidden algebra will recognise that a *basic* cosignature in our sense corresponds to the one of a hidden signature (cf., e.g., [15]) in which an operation can have at most one argument of hidden sort. The notion of a cosignature we are using slightly generalises the notion of a signature in hidden algebra by allowing a cooperation to have values of different sorts depending on its argument. In particular, the value of a cooperation can be sometimes of hidden sort and sometimes of observable sort. An important example for this phenomenon is the predecessor function $P : \mathbb{N} \to 1 + \mathbb{N}$ from Example 7(2).

For basic cosignatures Σ, the connection between Σ-coalgebras and hidden algebras for the “corresponding” hidden signature has been made precise in [4] where an isomorphism between the category of Σ-coalgebras and the category of corresponding hidden algebras is established.

4.2. Complete sets of cooperations

If we instantiate Definition 5 of an observational coalgebra to the case of the more concrete Σ-coalgebras we obtain our notion of a complete set of cooperations.

Definition 17. Let X be a set, let $\mathcal{S}$ be a set of sorts and Σ an $\mathcal{S}$-sorted cosignature. A set of cooperations $\{f_\sigma : X \to X_{a(\sigma)}\}_{\sigma \in \Sigma}$ is called *complete* for X if the final map $\varphi : X \to \Omega_{G_\Sigma}$ from the corresponding Σ-coalgebra $(X, \langle f_\sigma : \sigma \in \Sigma \rangle)$ into the final Σ-coalgebra $(\Omega_{G_\Sigma}, \omega_\Sigma)$ is injective:

$$\begin{tikzcd} X \arrow[r, dashed, "\exists ! \varphi"] \arrow[d, "\langle f_\sigma : \sigma \in \Sigma \rangle"'] & \Omega_{G_\Sigma} \arrow[d, "\omega_\Sigma"] \\ G_\Sigma X \arrow[r, dashed, "G_\Sigma \varphi"'] & G_\Sigma \Omega_{G_\Sigma} \end{tikzcd}$$

Examples of complete sets of cooperations can be found in Example 7 (2)–(7) above.

Example 18

(1) In Example 7(2) the set $\mathcal{S}$ of sorts consists only of the one-element set 1. The cooperation P has arity $(\epsilon, 1) + (\epsilon, \mathsf{H})$, where ϵ denotes the empty word, and $\{P\}$ is a complete set of cooperations for $\mathbb{N}$.

(2) Example 7(3) does not immediately give rise to a complete set of cooperations. We first have to split the given function $P_p : \mathbb{N} \to 1 + \{0, \ldots, p-1\} \times \mathbb{N}$ into two functions $P^1_p : \mathbb{N} \to 1 + \{0, \ldots, p-1\}$ and $P^2_p : \mathbb{N} \to 1 + \mathbb{N}$ by letting $P^1_p(0) = P^2_p(0) = * \in 1$ and by putting for all $n > 0$, $P^1_p(n) := n \bmod p$ and $P^2_p(n) := \lfloor \frac{n}{p} \rfloor$. It is now easy to see that $\{P^1_p, P^2_p\}$ is a complete set of cooperations for $\mathbb{N}$ with $a(P^1_p) = (\epsilon, 1) + (\epsilon, \{0, \ldots, p-1\})$ and $a(P^2_p) = (\epsilon, 1) + (\epsilon, \mathsf{H})$.

(3) In Example 7(4) the set $\mathcal{S}$ consists of the set A and the cooperations $h : A^{\mathbb{Z}} \to A$, $l : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ and $r : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ with arities (ϵ, A), (ϵ, H) and (ϵ, H), respectively, form a complete set of cooperations for $A^{\mathbb{Z}}$.

Remark 19. Equivalently, we could have defined complete sets of cooperations in the following way:

$$\{f_\sigma\}_{\sigma\in\Sigma} \text{ is complete for } X \text{ if } (X, \{f_\sigma\}_{\sigma\in\Sigma}) \text{ is strongly extensional.} \tag{1}$$

That the completeness condition in (1) is implied by the one in Definition 17 is an immediate consequence of Proposition 10. The converse direction can be proven using the observation that for any functor of the form G_Σ for some cosignature Σ the relation $\leftrightarrows_{G_\Sigma}$ is transitive.

4.3. *Example: completeness of* {head, even, odd}

As mentioned before, the notion of a cobasis from hidden algebra is closely related to our notion of a complete set of cooperations. In [15] it has been shown that there is a behavioural specification of infinite streams over some set A such that the set {head, even, odd} constitutes a cobasis. We now give a coalgebraic proof of the fact that

$$\{\mathrm{head} : A^\omega \to A, \mathrm{even} : A^\omega \to A^\omega, \mathrm{odd} : A^\omega \to A^\omega\}$$

is a complete set of cooperations for the set of A-streams, where for any infinite A-stream $\alpha = a_0a_1a_2a_3a_4a_5\ldots \in A^\omega$ we have

$$\mathrm{head}(\alpha) := a_0 \qquad \mathrm{even}(\alpha) := a_0a_2a_4\ldots \qquad \mathrm{odd}(\alpha) := a_1a_3a_5\ldots.$$

Definition 20. Let $A^{2^*} := \{t \mid t : 2^* \to A\}$ be the set of infinite binary A-labelled trees. For a tree $t \in A^{2^*}$ and a word $w \in 2^*$ we denote by t_w the tree given by $t_w(v) := t(vw)$ for all $v \in 2^*$.

In other words we code A-labelled infinite binary trees as functions $t : 2^* \to A$. Here nodes of a tree are identified with elements of 2^* in the usual way: the empty word ϵ corresponds to the root of the tree and if some $w \in 2^*$ corresponds to a node in the tree, then $0w$ and $1w$ correspond to the left and the right successor of this node, respectively. For any $w \in 2^*$ and any A-labelled tree $t : 2^* \to A$, the tree $t_w : 2^* \to A$ represents the tree that is obtained from t by taking w as the new root.

In the following, we will work with the binary coding of natural numbers.

Remark 21. We follow the convention that the most significant digit of the binary coding of a natural number is the leftmost digit, e.g., the natural number 13 is encoded as the sequence 1101.

Definition 22. We denote by $\mathrm{bin} : \mathbb{N} \to 2^*$ the function that maps a natural number to its representation in binary coding. Furthermore we denote by $\mathrm{nat} : 2^* \to \mathbb{N}$ the function that maps a binary code to the corresponding natural number. By convention we put $\mathrm{nat}(\epsilon) := 0$.

The following is a well-known fact from universal coalgebra (see, e.g., [18]).

Fact 23. Define $h : A^{2^*} \to A$ by $h(t) := t(\epsilon)$, $l : A^{2^*} \to A^{2^*}$ by $l(t) := t_0$ and $r : A^{2^*} \to A^{2^*}$ by $r(t) := t_1$. The set A^{2^*} together with the map $\langle h, l, r\rangle : A^{2^*} \to A \times A^{2^*} \times A^{2^*}$ form a final coalgebra for the functor $A \times \mathrm{Id} \times \mathrm{Id}$.

We now prove that {head, even, odd} is a complete set of cooperations.

Proposition 24. *Let $j : A^\omega \to A^{2^*}$ be the function that maps a stream τ to the binary tree $j(\tau)$ with*

$$j(\tau)(w) := \tau_{\mathrm{nat}(w)} \quad \textit{for all } w \in 2^*.$$

Then j is the unique coalgebra morphism from $(A^\omega, \langle \mathrm{head}, \mathrm{even}, \mathrm{odd}\rangle)$ into the final coalgebra $(A^{2^*}, \langle h, l, r\rangle)$.

Proof. We have to prove that the following diagram commutes:

$$\begin{tikzcd}
A^\omega \arrow[r, "j"] \arrow[d, "\langle \mathrm{head},\mathrm{even},\mathrm{odd}\rangle"'] & A^{2^*} \arrow[d, "\langle h,l,r\rangle"] \\
A \times A^\omega \times A^\omega \arrow[r, "\mathrm{id}\times j\times j"'] & A \times A^{2^*} \times A^{2^*}
\end{tikzcd}$$

Let $\tau \in A^\omega$ be a stream. Then $\mathrm{head}(\tau) = \tau_0 = \tau_{\mathrm{nat}(\epsilon)} = j(\tau)(\epsilon) = h(j(\tau))$. Furthermore for $w \in 2^*$ we get

$$l(j(\tau))(w) = j(\tau)_0(w) = j(\tau)(w0)$$
$$= \tau_{\mathrm{nat}(w0)} = \tau_{2*\mathrm{nat}(w)} = \mathrm{even}(\tau)_{\mathrm{nat}(w)} = j(\mathrm{even}(\tau))(w)$$

and

$$r(j(\tau))(w) = j(\tau)_1(w) = j(\tau)(w1)$$
$$= \tau_{\mathrm{nat}(w1)} = \tau_{2*\mathrm{nat}(w)+1} = \mathrm{odd}(\tau)_{\mathrm{nat}(w)} = j(\mathrm{odd}(\tau))(w).$$

Hence $l(j(\tau)) = j(\mathrm{even}(\tau))$ and $r(j(\tau)) = j(\mathrm{odd}(\tau))$ which finishes the proof that the above diagram commutes. Therefore j is the unique coalgebra morphism into the final coalgebra $(A^{2^*}, \langle h, l, r\rangle)$. □

Corollary 25. *The set* {head, even, odd} *is a complete set of cooperations for* A^ω.

Proof. This follows immediately from Proposition 24 and the fact that $j : A^\omega \to A^{2^*}$ is obviously injective. □

Remark 26. Note that the function j from Proposition 24 is not surjective: Only those trees $t \in A^{2^*}$ lie in the range of j for which we have $h(t') = h(l(t'))$ for all subtrees t' of t. This observation will be important for the {head, even, odd}-definition scheme in Section 6.

5. The proof principle

We now turn to the discussion of G_Σ-bisimulations and of the resulting Σ-proof principle. It follows from Proposition 10 that an observational Σ-coalgebra $(H, \langle f_\sigma : \sigma \in \Sigma\rangle)$ is strongly extensional w.r.t. G_Σ-bisimilarity, i.e., $\tau_1 \leftrightarroweq_{G_\Sigma} \tau_2$ implies $\tau_1 = \tau_2$ for all $\tau_1, \tau_2 \in X$. Let us first spell out the definition of a Σ-bisimulation.

Fact 27. Let $(H, \langle f_\sigma : \sigma \in \Sigma\rangle)$ be a Σ-coalgebra. A relation $R \subseteq H \times H$ is a Σ-bisimulation if for all $(\tau_1, \tau_2) \in R$ and $f_\sigma : H \to H_{\alpha_1} + \cdots + H_{\alpha_n}$ with basic arities $\alpha_1, \ldots, \alpha_n$ we have

(1) $f_\sigma(\tau_1) \in H_{\alpha_i}$ *iff* $f_\sigma(\tau_2) \in H_{\alpha_i}$ *for* $1 \le i \le n$,
(2) *if* $f_\sigma(\tau_1), f_\sigma(\tau_2) \in H_{\alpha_i}$ *and* $\alpha_i = (S_1 \ldots S_m, S)$ *with* $S \in \mathcal{S}$ *we have*

$$f_\sigma(\tau_1)(s_1, \ldots, s_m) = f_\sigma(\tau_2)(s_1, \ldots, s_m) \quad \text{for all } s_i \in S_i,\ 1 \le i \le m,$$

(3) *if* $f_\sigma(\tau_1), f_\sigma(\tau_2) \in H_{\alpha_i}$ *and* $\alpha_i = (S_1 \ldots S_m, \mathsf{H})$ *we have*

$$(f_\sigma(\tau_1)(s_1, \ldots, s_m), f_\sigma(\tau_2)(s_1, \ldots, s_m)) \in R \quad \text{for all } s_i \in S_i,\ 1 \le i \le m.$$

As observational coalgebras are strongly extensional we obtain the following Σ-coinduction proof principle for a set H that is equipped with a complete set of cooperations.

Proposition 28. *Let* Σ *be a cosignature and suppose* $\mathcal{O} = \{f_\sigma : \sigma \in \Sigma\}$ *is a complete set of cooperations for a set* H. *For all* $\tau_1, \tau_2 \in H$ *and all* Σ*-bisimulations* $R \subseteq H \times H$ *we have* $(\tau_1, \tau_2) \in R$ *implies* $\tau_1 = \tau_2$.

Proof. The claim follows from Proposition 10. □

The following proposition describes a special, slightly simpler case of the Σ-coinduction proof principle.

Proposition 29. *Let* Σ *be a cosignature, let* $\mathcal{O} = \{f_\sigma : \sigma \in \Sigma\}$ *be a complete set of cooperations for a set* H *and let* $\tau_1, \tau_2 \in H$. *Suppose for all cooperations* $f_\sigma : H \to H_{\alpha_1} + \cdots + H_{\alpha_n}$ *the following holds:*

(1) $f_\sigma(\tau_1) \in H_{\alpha_i}$ *iff* $f_\sigma(\tau_2) \in H_{\alpha_i}$ *for* $1 \le i \le n$
(2) *if* $f_\sigma(\tau_1), f_\sigma(\tau_2) \in H_{\alpha_i}$ *and* $\alpha_i = (S_1 \ldots S_m, T)$ *we have*

$$f_\sigma(\tau_1)(s_1, \ldots, s_m) = f_\sigma(\tau_2)(s_1, \ldots, s_m) \quad \text{for all } s_i \in S_i,\ 1 \le i \le m.$$

Then we can conclude that $\tau_1 = \tau_2$.

Proof. Given the assumptions of the proposition it is straightforward to see that the relation $\Delta_H \cup \{(\tau_1, \tau_2)\}$ is a Σ-bisimulation, where $\Delta_H \subseteq H \times H$ denotes the identity relation (the H-"diagonal"). Therefore the claim follows using Proposition 28. □

Remark 30. As mentioned in Section 4.1 basic cosignatures can be seen as a hidden algebra signature. From this perspective the Σ-coinduction proof principle is similar to the coinduction principle used in hidden algebra (see [15]).

We now turn to an example that should demonstrate that a good choice of a complete set of cooperations for a given set H can lead to relatively simple proofs by Σ-coinduction. Further applications of Σ-coinduction can be found in Section 6 (cf., Proposition 52 and Example 64 below).

5.1. The proof principle: an example

Consider the set $\mathbb{R}^\omega$ of streams of real numbers together with the complete set of cooperations $\{h, \Delta\}$ from Example 7(6). We will recall a bit of so-called stream calculus; see [17] for all details. Let $X = (0, 1, 0, 0, 0, \ldots)$. The convolution product $\sigma \times \tau$ of two streams σ and τ in $\mathbb{R}^\omega$ is given, for all $n \geq 0$, by

$$(\sigma \times \tau)(n) = \sum_{0 \leq k \leq n} \sigma(k) \cdot \tau(n-k)$$

The multiplicative inverse of τ is denoted by $1/\tau$ (which exists whenever $\tau(0) \neq 0$). As usual, σ/τ denotes $\sigma \times (1/\tau)$. We define the following so-called *falling powers* of X, for all $n \geq 0$, by

$$X^{\underline{n}} = X^n/(1-X)^{n+1}$$

As usual, we include the set of reals $\mathbb{R}$ into the set of streams $\mathbb{R}^\omega$ by the notational convention

$$r = (r, 0, 0, 0, \ldots)$$

Note that $\Delta X^{\underline{0}} = \Delta(1/(1-X)) = 0$ and

$$\Delta X^{\underline{n+1}} = X^{\underline{n}}$$

For $\sigma \in \mathbb{R}^\omega$ we define

$$r_n^\sigma = \left(\Delta^{(n)}\sigma\right)(0)$$

Now let

$$sum(\sigma) = r_0^\sigma \times X^{\underline{0}} + r_1^\sigma \times X^{\underline{1}} + r_2^\sigma \times X^{\underline{2}} + \cdots$$

Theorem 31. *For all $\sigma \in \mathbb{R}^\omega$,*

$$\sigma = sum(\sigma)$$

Proof. We show that

$$R = \{\, (\sigma,\ sum(\sigma)\,) \mid \sigma \in \mathbb{R}^\omega \,\}$$

is an $\{h, \Delta\}$-bisimulation. Clearly,

$$h(\sigma) = \sigma(0) = h(sum(\sigma))$$

Furthermore we have

$$\begin{aligned}
&\Delta\, sum(\sigma)\\
&= \Delta\left(r_0^\sigma \times X^{\underline{0}} + r_1^\sigma \times X^{\underline{1}} + r_2^\sigma \times X^{\underline{2}} + \cdots\right)\\
&= r_0^\sigma \times \Delta X^{\underline{0}} + r_1^\sigma \times \Delta X^{\underline{1}} + r_2^\sigma \times \Delta X^{\underline{2}} + \cdots\\
&= r_1^\sigma \times X^{\underline{0}} + r_2^\sigma \times X^{\underline{1}} + r_3^\sigma \times X^{\underline{2}} + \cdots\\
&= sum(\Delta\sigma)
\end{aligned}$$

where for the latter equality we use

$$r_{n+1}^\sigma = r_n^{\Delta\sigma}$$

As a consequence, we have

$$(\,\Delta\sigma,\ \Delta\, sum(\sigma)\,) = (\,\Delta\sigma,\ sum(\Delta\sigma)\,) \in R$$

This proves that R is an $\{h, \Delta\}$-bisimulation. $\square$

The theorem above is already present in [17, Theorem 11.1]. The reader is invited to compare the proof there with the present one. (Giving away the clue, the present one is quite a bit simpler.)

6. The definition scheme

In this section, we develop a scheme for corecursively defining constants and functions using a given complete set of cooperations. This scheme is closely related to similar ones in the hidden algebra community, in particular to the one in [3]. As we have already pointed out in Section 1 we believe that the novelty of our scheme lies in the use of coalgebraic techniques and in the fact that our scheme allows for the simultaneous definition of several functions. For an example of the latter phenomenon the reader is referred to Example 62.

6.1. The general idea

In order to provide the reader with a good feeling for the ideas that underlie the following quite technical section, we start the discussion of the definition scheme by looking at some examples.

Example 32. Consider the set of bitstreams 2^ω, i.e., of streams over the two-element set $\{0, 1\}$. In Section 4.3 we saw that the set $\{\text{head}, \text{even}, \text{odd}\}$ is a complete set of cooperations for 2^ω. Using our definition scheme we will be able to define functions of type $(2^\omega)^n \to 2^\omega$ for $n \in \mathbb{N}$ (for $n = 0$ we obtain a constant).

As a concrete example, we treat the so-called *Thue–Morse sequence* (cf., [1]) $\mathrm{TM} = t_0 t_1 t_2 \ldots$ with $t_n = s_2(n) \bmod 2$, where $s_2(n)$ denotes the sum of the digits of the binary representation of n. In our scheme for the complete set of cooperations $\{\text{head}, \text{even}, \text{odd}\}$ we define TM by specifying the constant TM and an auxiliary function $\text{inv} : 2^\omega \to 2^\omega$ that computes the inverse of a given bitstream, i.e., 0 is replaced by 1 and 1 is replaced by 0. The specification consists of the following equations:

$$
\begin{aligned}
F_{\text{head}}(\text{inv}(x)) &= 1 - F_{\text{head}}(x) & \qquad F_{\text{head}}(\mathrm{TM}) &= 0\\
F_{\text{even}}(\text{inv}(x)) &= \text{inv}(F_{\text{even}}(x)) & F_{\text{even}}(\mathrm{TM}) &= \mathrm{TM}\\
F_{\text{odd}}(\text{inv}(x)) &= \text{inv}(F_{\text{odd}}(x)) & F_{\text{odd}}(\mathrm{TM}) &= \text{inv}(\mathrm{TM})
\end{aligned}
$$

More abstractly speaking, for any function or constant in $\Delta = \{\mathrm{TM}, \text{inv}\}$ we specify how it behaves under application of the cooperations head, even and odd. We will come back to the example later and show that the equations above indeed define the Thue–Morse sequence and the inverse function.

The previous example concerned cooperations for a *basic* cosignature, i.e., the cooperations are all of basic arity either of the form $(S_1 \cdots S_l, \mathsf{H})$ or $(S_1 \cdots S_l, S)$ for $l \in \mathbb{N}, S_1, \ldots, S_l, S \in \mathcal{S}$. When dealing with arbitrary sets of cooperations the definition scheme can involve some case distinctions as the following simple example demonstrates.

Example 33. Consider the set of extended natural numbers $\bar{\mathbb{N}}$ together with the cooperation $\bar{P} : \bar{\mathbb{N}} \to 1 + \bar{\mathbb{N}}$ where $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. In Example 2 we remarked that $(\bar{\mathbb{N}}, \bar{P})$ is a final $1 + \mathrm{Id}$-coalgebra and thus $\{\bar{P}\}$ is a complete set of cooperations for $\bar{\mathbb{N}}$. Suppose now that we want to define multiplication $\text{mult} : \bar{\mathbb{N}}^2 \to \bar{\mathbb{N}}$ and addition $\text{add} : \bar{\mathbb{N}}^2 \to \bar{\mathbb{N}}$ using our scheme. Similar to what we did in the previous example we have to specify the behaviour under taking the predecessor for each function symbol in $\Delta = \{\text{mult}, \text{add}\}$. The equations involve case distinctions as to whether the variables "behave" like 0 or like a natural number greater than 0. This will be encoded by so-called *behaviour patterns*. In the example we assign to a variable x of sort H a behaviour pattern $\Gamma(x) : \{\bar{P}\} \to \mathbb{N}$ with either $\Gamma(x)(\bar{P}) = 1$ (which should be read as "$\bar{P}(x)$ is contained in the first component of the coproduct") or $\Gamma(x)(\bar{P}) = 2$ (to be read as "$\bar{P}(x)$ is contained in the second component of the coproduct"). The following equations specify addition and multiplication:

$$
F_{\bar{P}}(\text{add}(x, y)) = \begin{cases}
\text{add}(F_{\bar{P}}(x), y) & \text{if } \Gamma(x)(\bar{P}) = 2, \Gamma(y)(\bar{P}) = 1 & (\text{“}x \neq 0\text{”})\\
\text{add}(x, F_{\bar{P}}(y)) & \text{if } \Gamma(x)(\bar{P}) = 1, \Gamma(y)(\bar{P}) = 2 & (\text{“}y \neq 0\text{”})\\
\text{add}(F_{\bar{P}}(x), y) & \text{if } \Gamma(x)(\bar{P}) = \Gamma(y)(\bar{P}) = 2 & (\text{“}x, y \neq 0\text{”})\\
* & \text{if } \Gamma(x)(\bar{P}) = \Gamma(y)(\bar{P}) = 1 & (\text{“}x, y = 0\text{”})
\end{cases}
$$

$$F_{\bar{P}}(\mathrm{mult}(x,y)) = \begin{cases} * & \text{if } \Gamma(x)(\bar{P}) = 2, \Gamma(y)(\bar{P}) = 1 \\ & \quad (\text{“}x \neq 0, y = 0\text{”}) \\ * & \text{if } \Gamma(x)(\bar{P}) = 1, \Gamma(y)(\bar{P}) = 2 \\ & \quad (\text{“}y \neq 0, x = 0\text{”}) \\ \mathrm{add}(F_{\bar{P}}(x), \mathrm{mult}(F_{\bar{P}}(y), x)) & \text{if } \Gamma(x)(\bar{P}) = \Gamma(y)(\bar{P}) = 2 \\ & \quad (\text{“}x, y \neq 0\text{”}) \\ * & \text{if } \Gamma(x)(\bar{P}) = \Gamma(y)(\bar{P}) = 1 \\ & \quad (\text{“}x, y = 0\text{”}) \end{cases}$$

Example 34. For another example consider the set A^∞ of all finite or infinite lists over some set A. A complete set of cooperations is provided by the set $\{\mathrm{head} : A^\infty \to 1 + A, \mathrm{tail} : A^\infty \to 1 + A^\infty\}$ where $\mathrm{head}(\sigma)$ and $\mathrm{tail}(\sigma)$ for a non-empty list σ are defined as usual and $\mathrm{head}(\epsilon) = \mathrm{tail}(\epsilon) = * \in 1$, where ϵ denotes the empty list. We will be able to specify a function $\mathrm{zip} : (A^\infty)^2 \to A^\infty$ as follows:

$$F_{\mathrm{head}}(\mathrm{zip}(x,y)) = \begin{cases} F_{\mathrm{head}}(x) & \text{if } \mathrm{head}(x) = 2, \mathrm{head}(y) = 2 \\ F_{\mathrm{head}}(x) & \text{if } \mathrm{head}(x) = 2, \mathrm{head}(y) = 1 \\ F_{\mathrm{head}}(y) & \text{if } \mathrm{head}(x) = 1, \mathrm{head}(y) = 2 \\ * & \text{if } \mathrm{head}(x) = \mathrm{head}(y) = 1 \end{cases}$$

$$F_{\mathrm{tail}}(\mathrm{zip}(x,y)) = \begin{cases} F_{\mathrm{tail}}(x) & \text{if } \mathrm{tail}(x) = 2, \mathrm{tail}(y) = 1 \\ F_{\mathrm{tail}}(y) & \text{if } \mathrm{tail}(x) = 1, \mathrm{tail}(y) = 2 \\ \mathrm{zip}(y, F_{\mathrm{tail}}(x)) & \text{if } \mathrm{tail}(x) = 2, \mathrm{tail}(y) = 2 \\ * & \text{if } \mathrm{tail}(x) = 1, \mathrm{tail}(y) = 1 \end{cases}$$

Note that in this example we wrote $\mathrm{head}(x) = 1$ instead of saying that the behaviour pattern $\Gamma(x)$ of x has the property that $\Gamma(x)(\mathrm{head}) = 1$. Although the latter formulation would be a more exact account of what we are doing in the general formulation of the definition scheme below, we opted for the first notation as we feel that it makes the example more readable.

We will come back to the examples in order to motivate the definitions in this section. We are now turning to the description of the general definition scheme. Our definition scheme for H-constants and H-functions generalises the scheme that has been presented in [18] for the case that H is the set of infinite binary A-labelled trees. The scheme extends the one presented in [12] by allowing sets of cooperations for arbitrary cosignature, whereas the scheme presented in [12] worked for *basic* cosignatures only.

In the remainder of this section we assume that we are given

- a collection $\mathcal{S}$ of sets (“observable sorts”) and a hidden sort H,
- a set H, a finite cosignature Σ and a complete set of cooperations $\mathcal{O} = \{f_\sigma : H \to H_{a(\sigma)}\}_{\sigma \in \Sigma}$ for H which constitute a Σ-coalgebra $\mathbb{X} = (H, \langle f_\sigma : \sigma \in \Sigma \rangle)$,
- a set Δ of function symbols for the functions that we want to define; we write $\Delta_i \subseteq \Delta$ for the set of function symbols in Δ with $i \in \mathbb{N}$ arguments.

In order to be able to formulate what a well-formed definition of H-constants and functions is, we have to introduce some syntax.

6.1.1. The terms

We first define the set $\mathcal{SE}$ of state equation terms, the set $\mathcal{E}$ of equation terms and the set $\mathcal{E}_{\mathrm{res}}$ of restricted equation terms. These terms are sorted, i.e., we write $t : S$ to indicate that t is a term of sort $S \in \mathcal{S} \cup \{\mathsf{H}\}$. In our scheme, we are allowed to freely use “help functions” of observable sort.

Definition 35. For a set $\mathcal{S}$ of sorts we define the set of help functions by putting $\mathtt{Help}_{\mathcal{S}} := \{h \mid h \text{ is a function of type } S_1 \times \cdots \times S_j \to T \text{ for some } j \in \mathbb{N} \text{ and some } S_1, \ldots, S_j, T \in \mathcal{S}\}$.

The terms will be generated over a set of variables that not only have a sort $S \in \mathcal{S} \cup \{\mathsf{H}\}$ associated with them but in addition a function Γ that encodes the behaviour of a variable under the cooperations in Σ.

Definition 36. Let $\mathcal{S}$ be a set of sorts and let Σ be an $\mathcal{S}$-sorted cosignature. Furthermore let Δ be a set of constants and function symbols and let $\mathcal{X} = (X_S)_{S\in\mathcal{S}\cup\{\mathsf{H}\}}$ be a sorted sets of variables together with a function $\Gamma : X_{\mathsf{H}} \to (\Sigma \to \mathbb{N})$ such that

(i) for all $\sigma \in \Sigma$ with $a(\sigma) = \alpha_1 + \cdots + \alpha_m$ we have $\Gamma(x)(\sigma) \in \{1, \ldots, m\}$ and
(ii) for all Σ-indexed families of natural numbers $\{i_\sigma\}_{\sigma\in\Sigma}$ such that $i_\sigma \in \{1, \ldots, k\}$ if $a(\sigma) = \alpha_1 + \cdots + \alpha_k$ the set $\{x \in X_{\mathsf{H}} \mid \Gamma(x)(\sigma) = i_\sigma \text{ for all } \sigma \in \Sigma\}$ is infinite for all $i \in I$.

We call a function $\nu : \Sigma \to \mathbb{N}$ a *behaviour pattern*. We define the set $\mathcal{E}$ of equation terms as follows:

$$
\begin{aligned}
\mathcal{E} \ni s ::= {} & x : S, x \in X_S, S \in \mathcal{S} \cup \{\mathsf{H}\} \mid \underline{s} : S, s \in S, S \in \mathcal{S} \mid \underline{\tau} : \mathsf{H}, \tau \in H \\
& \mid F_\sigma(x) : S_1 \times \cdots \times S_n \to T,\ \sigma \in \Sigma, \\
& \quad a(\sigma) = \alpha_1 + \cdots + \alpha_m, \Gamma(x)(\sigma) = i,\ \alpha_i = (S_1 \ldots S_n, T), x \in X_{\mathsf{H}} \\
& \mid F_\sigma(\underline{\tau}) : S_1 \times \cdots \times S_n \to T, \tau \in H \text{ with } f_\sigma(\tau) : S_1 \times \cdots \times S_n \to T \\
& \mid F_\sigma(t) : (S_{1,1} \times \cdots \times S_{1,n_1} \to T_1) + \cdots + (S_{k,1} \times \cdots \times S_{k,n_k} \to T_k), \\
& \quad \sigma \in \Sigma, a(\sigma) = (S_{1,1} \ldots S_{1,n_1}, T_1) + \cdots + (S_{k,1} \ldots S_{k,n_k}, T_k), t : \mathsf{H} \\
& \mid h : S_1 \times \cdots \times S_l \to S \in \mathtt{Help}_{\mathcal{S}} \mid g : (\mathsf{H})^n \to \mathsf{H},\ g \in \Delta_n, n \in \mathbb{N} \\
& \mid t(t_1, \ldots, t_l) : S,\ t : S_1 \times \cdots \times S_l \to S, t_i : S_i \text{ for } 1 \le i \le l
\end{aligned}
$$

and by letting

- the set $\mathcal{E}_{\text{res}} \subseteq \mathcal{E}$ of restricted equation terms to consist exactly of those terms in $\mathcal{E}$ in which for every $\sigma \in \Sigma$ the symbol F_σ is applied to variables only, and
- the set $\mathcal{SE} \subseteq \mathcal{E}$ of state equation terms to consist exactly of those terms in $\mathcal{E}$ that do not contain any occurence of a $g \in \Delta$, $h \in \mathtt{Help}_{\mathcal{S}}$ or of some $\underline{\tau}$ with $\tau \in H$.

We write $t(x_1 : S_1, \ldots, x_n : S_n)$ in order to indicate that t is a term with variables contained in $\{x_1 : S_1, \ldots, x_n : S_n\}$. Finally we put

$$\mathcal{T} \ni t ::= \underline{\tau}, \tau \in H \mid g(t_1, \ldots, t_n), g \in \Delta_n, n \in \mathbb{N}.$$

Note that the set $\mathcal{E}$ of equation terms can be seen as the set of terms for an algebraic signature consisting of the function symbols F_σ for $\sigma \in \Sigma$, the function symbols $g \in \Delta$ and the constants $\underline{s}$ for $s \in S, S \in \mathcal{S}$, and $\underline{\tau}$ for $\tau \in H$. The reason why our definition looks rather involved is that we have to ensure that all terms are correctly sorted and that the assignment of sorts respects the behaviour pattern of the variables that occur in a term.

All equations will be written using sorted terms in $\mathcal{E}$. Right-hand sides of behavioural differential equations will be restricted to terms in $\mathcal{E}_{\text{res}}$ and state equations $\mathcal{SE}$ will be used to describe H as a subcoalgebra of a final coalgebra (this explains why no symbols $g \in \Delta$ occur in a term in $\mathcal{SE}$). Finally the terms in $\mathcal{T}$ will be used as the carrier of a term coalgebra – the unique map from the term coalgebra into the final coalgebra will yield the intended interpretation of the terms in $\mathcal{T}$. Note that $\mathcal{T}$ can in fact also be seen as a subset of $\mathcal{E}_{\text{res}}$. The sort of some term t in $\mathcal{T}$, however, would always be H. Therefore we decided to write terms in $\mathcal{T}$ without any sorting information.

Example 37. The "behaviour pattern" of a variable $x \in X_{\mathsf{H}}$ will allow us to define functions by case distinction, similar to what we did in Examples 33 and 34. The definition of add in Example 33 will be written down in four separate equations – one equation for any possible behaviour pattern of the variables x and y. For writing down the equation treating the first case ($x \neq 0, y = 0$) we take two variables x and y such that $\Gamma(x)(\bar{P}) = 2$ and $\Gamma(y)(\bar{P}) = 1$ and we write the equation

$$F_{\bar{P}}(\text{add}(x, y)) = \text{add}(F_{\bar{P}}(x), y).$$

Similarly we treat the remaining three cases for defining add.

From now on we assume that we are working with a given set of $\mathcal{S}$-sorted variables together with a function Γ satisfying conditions (i) and (ii) from the previous definition. We also define what the behaviour pattern of a state of some Σ-coalgebra is.

Definition 38. Let $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma\rangle)$ be a Σ-coalgebra. The behaviour pattern $\Gamma_{\mathbb{Y}}(y) : \Sigma \to \mathbb{N}$ of a state y is defined by

$$\Gamma_{\mathbb{Y}}(y)(\sigma) = i \quad \text{if} \quad o_\sigma(y) \in Y_{\alpha_i},$$

for every $\sigma \in \Sigma$, $a(\sigma) = \alpha_1 + \cdots + \alpha_n$. In case there is no danger of ambiguity we simply write $\Gamma(y)$ instead of $\Gamma_{\mathbb{Y}}(y)$.

We will now specify how equation terms are interpreted over Σ-coalgebras. First we have to introduce the notion of an admissible valuation.

Definition 39. Let $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma \rangle)$ be a Σ-coalgebra and let V be a set of sorted variables from $\mathcal{X}$. A *variable assignment on* V is a function β that assigns to each variable $x \in V$ of sort $T \in \mathcal{S} \cup \{\mathsf{H}\}$ an element $s \in S$ if $T = S \in \mathcal{S}$ or a state $y \in Y$ if $T = \mathsf{H}$. A variable assignment β is called *admissible* if for all variables $x \in V$ of sort H and all $\sigma \in \Sigma$, $\Gamma(x)(\sigma) = i$ implies that $o_\sigma(\beta(x)) \in Y_{\alpha_i}$, where $a(\sigma) = \alpha_1 + \cdots + \alpha_n$.

In other words, a variable assignment β is admissible if for all variables x of sort H we have that x and $\beta(x)$ have the same behaviour pattern.

Example 40. Consider again Example 33 and let x be a variable of sort $\bar{\mathbb{N}}$ such that $\Gamma(x)(\bar{P}) = 1$. Then for any $1 + \mathrm{Id}$ coalgebra $(Y, o_{\bar{P}} : Y \to 1 + Y)$ and any variable assignment β on $\{x\}$ we have β is admissible iff $o_{\bar{P}}(\beta(x)) = * \in 1$, i.e., iff $\beta(x) \in Y$ "behaves like" the zero element $0 \in \bar{\mathbb{N}}$.

We introduce admissible variable assignments in order to be able to properly define the interpretation of a given term. For admissible valuations the interpretation of a term is defined as usually. The interpretation of a term with respect to a non-admissible variable assignment will be a certain default value $\perp$ (which should be read as "undefined").

Definition 41. Let $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma \rangle)$ be a Σ-coalgebra and suppose that for every $g \in \Delta_m$ there is an operation $g^{\mathbb{Y}} : Y^m \to Y$. For every term $t(x_1, \ldots, x_n) \in \mathcal{E}$ and every admissible variable assignment α on $\{x_1, \ldots, x_n\}$ we define by induction on t its interpretation $(t[\alpha])^{\mathbb{Y}}$ as follows:

$$\begin{aligned}
(x[\alpha])^{\mathbb{Y}} &:= \alpha(x) \qquad (\underline{s}[\alpha])^{\mathbb{Y}} := s \in S \qquad (\underline{\tau}[\alpha])^{\mathbb{Y}} := \tau \in H \\
(h[\alpha])^{\mathbb{Y}} &:= h \qquad\quad (g[\alpha])^{\mathbb{Y}} := g^{\mathbb{Y}} \\
(F_\sigma(t)[\alpha])^{\mathbb{Y}} &:= o_\sigma((t[\alpha])^{\mathbb{Y}}) \\
(t(t_1, \ldots, t_n)[\alpha])^{\mathbb{Y}} &:= (t[\alpha])^{\mathbb{Y}}((t_1[\alpha])^{\mathbb{Y}}, \ldots, (t_n[\alpha])^{\mathbb{Y}}).
\end{aligned}$$

If α is a variable assignment that is not admissible we put $(t[\alpha])^{\mathbb{Y}} := \perp$. Where necessary, we explicitly mention the interpretations of the function symbols in Δ and write $(t[\alpha])^{(\mathbb{Y}, \{g^{\mathbb{Y}}\}_{g \in \Delta})}$ for $(t[\alpha])^{\mathbb{Y}}$. Similarly we define the interpretation $(t[\alpha])^{\mathbb{Y}}$ of a term $t(x_1, \ldots, x_n) \in \mathcal{SE}$ on an arbitrary Σ-coalgebra. An *equation* is a pair of terms $e_1(x_1, \ldots, x_n), e_2(x_1, \ldots, x_n) \in \mathcal{E}$. We say (e_1, e_2) is satisfied in $\mathbb{Y}$ by an assignment α if $(e_1[\alpha])^{\mathbb{Y}} = (e_2[\alpha])^{\mathbb{Y}}$. We write $\mathbb{Y}, \alpha \models (e_1, e_2)$ if (e_1, e_2) is satisfied by α. Furthermore we write $x \mapsto t$ for the variable assignment that maps the variable x to the term t.

Definition 42. A *state equation* is a pair $e = (e_1, e_2)$ of terms $e_1(x : \mathsf{H}), e_2(x : \mathsf{H}) \in \mathcal{SE}$. Given a Σ-coalgebra $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma \rangle)$ we say that e is satisfied at a state y if $\mathbb{Y}, (x \mapsto y) \models (e_1, e_2)$, i.e., if $(e_1[x \mapsto y])^{\mathbb{Y}} = (e_2[x \mapsto y])^{\mathbb{Y}}$. We write $y \models e$ if e is satisfied at y and we write $\mathbb{Y} \models e$ if $y \models e$ for all $y \in Y$.

Example 43. Let $(e_1(x), e_2(x))$ be a state equation. It is a consequence of our definition that a coalgebra state y trivially satisfies $(e_1(x), e_2(x))$ if y has a behaviour pattern different from the behaviour pattern of x. Consider the coalgebra $(\bar{\mathbb{N}}, \bar{P})$ from Example 33 and the state equation $(F_{\bar{P}}(x), x)$ with some variable x of sort $\bar{\mathbb{N}}$ such that $\Gamma(x)(\bar{P}) = 2$. Then clearly $0 \models (F_{\bar{P}}(x), x)$ because $(F_{\bar{P}}(x)[x \mapsto 0])^{(\bar{\mathbb{N}}, \bar{P})} = (x[x \mapsto 0])^{(\bar{\mathbb{N}}, \bar{P})} = \perp$, but $n \not\models (F_{\bar{P}}(x), x)$ for all $n \in \mathbb{N}$ such that $n > 0$.

We will use the fact that Σ-coalgebra morphisms preserve state equations: if f is a coalgebra morphism and e is some state equation satisfied at a state x then e is also satisfied at $f(x)$. This is the content of the following two lemmas.

Lemma 44. *Let $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma \rangle)$ and $\mathbb{Y}' = (Y', \langle o'_\sigma : \sigma \in \Sigma \rangle)$ be Σ-coalgebras and let $f : \mathbb{Y} \to \mathbb{Y}'$ be a Σ-coalgebra morphism. Let $y \in Y$ and let $\sigma \in \Sigma$ with $a(\sigma) = \alpha_1 + \cdots + \alpha_n$. The following holds:*

(i) *$o_\sigma(y) \in Y_{\alpha_i}$ iff $o'_\sigma(f(y)) \in Y'_{\alpha_i}$ for all $i \in \{1, \ldots, n\}$,*
(ii) *if $o_\sigma(y) \in Y_{\alpha_i}$ with $\alpha_i = (S_1 \cdots S_n, S)$ and $S \neq \mathsf{H}$ then for all $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ we have*

$$o_\sigma(y)(s_1, \ldots, s_n) = o'_\sigma(f(y))(s_1, \ldots, s_n), \quad \textit{and}$$

(iii) *if* $o_\sigma(y) \in Y_{\alpha_i}$ *with* $\alpha_i = (S_1 \cdots S_n, \mathsf{H})$ *then for all* $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ *we have*

$$f(o_\sigma(y)(s_1, \ldots, s_n)) = o'_\sigma(f(y))(s_1, \ldots, s_n).$$

Proof. The claim can be easily proven by spelling out the definitions of a Σ-coalgebra morphism. $\square$

As a consequence we get that behaviour patterns and state equations are "preserved" under coalgebra morphisms.

Lemma 45. *Let* $\mathbb{Y}_1 = (Y_1, \langle o^1_\sigma : \sigma \in \Sigma \rangle)$ *and* $\mathbb{Y}_2$ *be* Σ*-coalgebras, let* $f : \mathbb{Y}_1 \to \mathbb{Y}_2$ *be a* Σ*-coalgebra morphism and let* $e = (e_1, e_2)$ *be a state equation. Then for all* $y \in Y_1$ *we have*

$$\Gamma_{\mathbb{Y}_1}(y)(\sigma) = \Gamma_{\mathbb{Y}_2}(f(y))(\sigma) \quad \textit{for all } \sigma \in \Sigma \tag{2}$$

$$y \models e \;\Rightarrow\; f(y) \models e. \tag{3}$$

Proof. The first item of Lemma 44 shows that (2) holds. In order to prove (3) note that (2) implies that $(e_1[x \mapsto y])^{\mathbb{Y}_1} = \perp$ iff $(e_2[x \mapsto f(y)])^{\mathbb{Y}_2} = \perp$. If $(x \mapsto y)$ is an admissible valuation one can use items (ii) and (iii) of Lemma 44 in order to show by a straightforward induction on the term structure that for all terms $t(x : \mathsf{H}) \in \mathcal{SE}$ the following holds

$$\begin{aligned} f\left((t[x \mapsto y])^{\mathbb{Y}_1}\right) &= \left(t[x \mapsto f(y)]^{\mathbb{Y}_2}\right) \text{ if } t : \mathsf{H} \\ (t[x \mapsto y])^{\mathbb{Y}_1} &= (t[x \mapsto f(y)])^{\mathbb{Y}_2} \text{ if } t \text{ is of observable sort.} \end{aligned}$$

This clearly implies the claim. $\square$

In order to be able to use the fact that H together with the set of cooperations $\{f_\sigma : \sigma \in \Sigma\}$ is (isomorphic to) a subcoalgebra (U, γ_U) of the final Σ-coalgebra, we have to concretely describe (U, γ_U) using state equations and behaviour patterns: if we characterise (U, γ_U) by a set of state equations E and a set of behaviour patterns $\mathcal{B}$, we know that (U, γ_U) and consequently also $(H, \langle f_\sigma : \sigma \in \Sigma \rangle)$ is relatively final amongst all Σ-coalgebras that validate the state equations in E.

Definition 46. Let $\mathbb{F} = (\Omega_\Sigma, \langle \omega_\sigma : \sigma \in \Sigma \rangle)$ be the final Σ-coalgebra and let $P \subseteq \Omega_\Sigma$ be a subset of Ω_Σ. We denote by $\Box\mathbb{P} = (\Box P, \langle \omega_\sigma^{\Box P} : \sigma \in \Sigma \rangle)$ the largest subcoalgebra of $\mathbb{F}$ that is contained in P.

The well definedness of $\Box\mathbb{P}$ follows from the fact that for any $P \subseteq \Omega_\Sigma$ the largest subcoalgebra of $\mathbb{F}$ contained in P exists (cf., e.g., [9, Theorem 4.7]).

Definition 47. Let Σ be a cosignature and let $\mathcal{O} = \{f_\sigma \mid \sigma \in \Sigma\}$ be a complete set of Σ-cooperations for H. We say that a set of state equations E together with a set of behaviour patterns $\mathcal{B}$ *completely specifies* $(H, \langle f_\sigma : \sigma \in \Sigma \rangle)$ if

$$\Box\mathbb{P}_{E,\mathcal{B}} \cong (H, \langle f_\sigma : \sigma \in \Sigma \rangle)$$

where $P_{E,\mathcal{B}} := \{y \in \Omega_\Sigma \mid \forall e \in E.\ y \models e \text{ and } \Gamma_{\mathbb{F}}(y) \in \mathcal{B}\}$. In this case we call $(\mathcal{O}, E, \mathcal{B})$ a *complete* (Σ*-*)*specification of* H. We call $(\mathcal{O}, E, \mathcal{B})$ finite if $\mathcal{O}$, E and $\mathcal{B}$ are finite.

Example 48. A complete specification of the set A^∞ together with the complete set of cooperations {head, tail} as described in Example 34 looks as follows: $\mathcal{O} = \{\text{head}, \text{tail}\}$, $E = \emptyset$ and $\mathcal{B} = \{\nu_1, \nu_2\}$ where $\nu_1(\text{head}) = \nu_1(\text{tail}) = 1$ and $\nu_2(\text{head}) = \nu_2(\text{tail}) = 2$. In this case the set $\mathcal{B}$ of behaviour patterns expresses that either both the $\text{head}(\sigma) = *$ and $\text{tail}(\sigma) = *$ (if σ is the empty list ϵ) or $\text{head}(\sigma) \in A$ and $\text{tail}(\sigma) \in A^\infty$.

Lemma 49. *Let* $(\mathcal{O}, E, \mathcal{B})$ *be a complete* Σ*-specification of* H *and let* $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma \rangle)$ *be a* Σ*-coalgebra. If* $\mathbb{Y} \models e$ *for all* $e \in E$ *and* $\Gamma_{\mathbb{Y}}(y) \in \mathcal{B}$ *for all* $y \in Y$*, then there exists a unique* Σ*-coalgebra morphism* $\iota_{\mathbb{Y}} : Y \to H$.

Proof. Let $\varphi : Y \to \Omega_\Sigma$ be the unique Σ-coalgebra morphism into the final Σ-coalgebra $\mathbb{F}$. It follows from Lemma 45 that $\mathbb{Y} \models e$ for all $e \in E$ and $\Gamma_{\mathbb{Y}}(y) \in \mathcal{B}$ for all $y \in Y$ implies $\text{range}(\varphi) \subseteq P_E$. As $\text{range}(\varphi)$ is a subcoalgebra of Ω we get $\text{range}(\varphi) \subseteq \Box\mathbb{P}_E$. The existence of ι follows now from Proposition 12. $\square$

6.1.2. The differential equations

We now have introduced the necessary terminology in order to be able to state the main definition of this section. This definition involves the notion of an equation being provable in a restricted version of conditional equational logic. We do not want to spell out this notion, instead, the reader is referred to the brief overview in [19, Section 7.3] and references therein.

Definition 50. Let $(\mathcal{O}, E, \mathcal{B})$ be a complete, finite specification of H. A *well-behaved system of behavioural differential equations* for $(\mathcal{O}, E, \mathcal{B})$ and Δ is a set Spec which contains for every $g \in \Delta_n$, for any family of behaviour patterns $\overline{\nu} := \{\nu_j : \Sigma \to \mathbb{N}\}_{j\in\{1,\ldots,n\}}$ with $\nu_j \in \mathcal{B}$ for all $j \in \{1, \ldots, n\}$ and for any $\sigma \in \Sigma$ an equation $e^g_\sigma(\overline{\nu})$ of the form

$$(F_\sigma(g(x_1 : \mathsf{H}, \ldots, x_n : \mathsf{H})))\,(y_1 : S_1, \ldots, y_l : S_l) = t^g_\sigma(\overline{\nu})(x_1 : \mathsf{H}, \ldots, x_n : \mathsf{H}, y_1 : S_1, \ldots, y_l : S_l) : T,$$

together with a natural number

$$\gamma(g(\overline{\nu}))(\sigma) := k \in \{1, \ldots, m\},$$

where $\bar{x} := \{x_j : \mathsf{H}\}_{j\in\{1,\ldots,n\}}$ consists of pairwise distinct variables such that $\Gamma(x_j) = \nu_j$ for $j \in \{1, \ldots, n\}$, $t^g_\sigma(\overline{\nu}) : T$ is a term in $\mathcal{E}_{\mathrm{res}}$, σ has arity $a(\sigma) = \alpha_1 + \cdots + \alpha_m$ and $\alpha_k = (S_1 \ldots S_l, T)$. The resulting behaviour pattern $\gamma(g(\overline{\nu})) : \Sigma \to \mathbb{N}$ has to be an element of $\mathcal{B}$.

Furthermore we require that for all $(e_1(x : \mathsf{H}), e_2(x : \mathsf{H})) \in E$ and all terms $g(x_1, \ldots, x_n)$ with $g \in \Delta$ such that $\gamma(g(\overline{\nu})) = \Gamma(x)$ with $\nu_i := \Gamma(x_i)$ for all $i \in \{1, \ldots, n\}$ the following conditional equation is provable in conditional equational logic:

$$(e_1[x := g(x_1, \ldots, x_n)] = e_2[x := g(x_1, \ldots, x_n)]) \Leftarrow E(x_1, \ldots, x_n) \cup \mathsf{Equ}_{\overline{\nu}}, \tag{4}$$

where

$$E(x_1, \ldots, x_n) := \{(e_1[x := x_i], e_2[x := x_i]) \mid 1 \le i \le n,\ \ (e_1(x : \mathsf{H}), e_2(x : \mathsf{H})) \in E,\ \ \Gamma(x) = \Gamma(x_i)\},$$

and where $\mathsf{Equ}_{\overline{\nu}} := \{e^g_\sigma(\overline{\nu}) \in \mathrm{Spec} \mid g \in \Delta, \sigma \in \Sigma\}$.

Now that we know what a well-behaved system of equations is, we also want to see what a solution of these equations looks like.

Definition 51. A solution of a well-behaved system of behavioural differential equations Spec is a family of functions $\{\hat{g}\}_{g\in\Delta}$ that contains for all $g \in \Delta$ a function $\hat{g} : H^n \to H$ such that for all equations

$$(F_\sigma(g(x_1 : \mathsf{H}, \ldots, x_n : \mathsf{H})))\,(y_1 : S_1, \ldots, y_l : S_l) = t^g_\sigma(\overline{\nu})(x_1 : \mathsf{H}, \ldots, x_n : \mathsf{H}, y_1 : S_1, \ldots, y_l : S_l) : T,$$

in Spec, for all $\tau_1, \ldots, \tau_n \in H$ with $\Gamma(\tau_j) = \nu_j$ for $j \in \{1, \ldots, n\}$ and for all $s_1 \in S_1, \ldots, s_l \in S_l$ we have

$$f_\sigma(\hat{g}(\tau_1, \ldots, \tau_n)) = \kappa_k\left(\lambda(s_1 : S_1). \ldots \lambda.(s_l : S_l).(t^g_\sigma(\overline{\nu})[y_j := \underline{s_j}][x_i \mapsto \underline{\tau_i}])^{\mathbb{X}}\right),$$

where $k = \gamma(g(\nu_1, \ldots, \nu_n))(\sigma)$ and $\kappa_k : H_{\alpha_k} \to H_{\alpha_1} + \cdots + H_{\alpha_m}$ is the canonical injection map into the k-th component with $a(\sigma) = \alpha_1 + \cdots + \alpha_m$.

Before we demonstrate that such a solution exists for any well-formed system of behavioural differential equations we demonstrate that a solution has to be necessarily unique.

Proposition 52. *If $\{\hat{g}\}_{g\in\Delta}$ and $\{g'\}_{g\in\Delta}$ are solutions of the well-behaved system of behavioural differential equations* Spec, *then for all $g \in \Delta$ we have $\hat{g} = g'$.*

Proof. In order to prove the proposition, we define a relation $R \subseteq H \times H$ that contains all pairs $(\hat{g}(\tau_1, \ldots, \tau_n), g'(\tau_1, \ldots, \tau_n))$ for all $g \in \Delta$ and all $\tau_i \in X$. The claim follows then by showing that R is a bisimulation.

We define the relation $R \subseteq H \times H$ by putting

$$R := \{(\hat{t}, t') \in H \times H \mid \exists t \in \mathcal{T}.\hat{t} = (t)^{\mathbb{X}, \{\hat{g}\}_{g\in\Delta}}, t' = (t)^{\mathbb{X}, \{g'\}_{g\in\Delta}}\}$$

The reader is invited to convince herself of the fact that for all $g \in \Delta$ and all $\tau_1, \ldots, \tau_n \in H$ we have $(\hat{g}(\tau_1, \ldots, \tau_n), g'(\tau_1, \ldots, \tau_n)) \in R$. Therefore for proving the claim of the proposition it suffices to demonstrate that R is a Σ-bisimulation.

In order to prove that R is a Σ-bisimulation we show by induction on t that for all $(\hat{t}, t') \in R$ the following holds true:

(i) for all $\sigma \in \Sigma$ we have $f_\sigma(\hat{t}) \in H_{\beta_j}$ iff $f_\sigma(t') \in H_{\beta_j}$ where $a(\sigma) = \beta_1 + \cdots + \beta_l$,
(ii) for all $\sigma \in \Sigma$ we have $f_\sigma(\hat{t}) = f_\sigma(t')$ if $f_\sigma(\hat{t})$ is a constant or function of observable sort, and
(iii) for all $\sigma \in \Sigma$ and for all $s_1 \in S_1, \ldots, s_l \in S_l$ we have

$$(f_\sigma(\hat{t})(s_1, \ldots, s_l), f_\sigma(t')(s_1, \ldots, s_l)) \in R_{n+1}$$

if $f_\sigma(\hat{t}) : S_1 \times \cdots \times S_l \to H$.

The base case of the induction, $t = \underline{\tau}$ for some $\tau \in H$, is trivial. For the induction step consider $t = g(t_1, \ldots, t_n) \in \mathcal{T}$ and let $\sigma \in \Sigma$. Using the induction hypothesis it is easy to see that the behaviour patterns of $\hat{t}_i := (t_i)^{\mathbb{X}, \{\hat{g}\}_{g \in \Delta}}$ and $t_i' := (t_i)^{\mathbb{X}, \{g'\}_{g \in \Delta}}$ coincide, i.e., $\Gamma(\hat{t}_i) = \Gamma(t_i')$ for all $i \in \{1, \ldots, n\}$. If we put $\nu_i := \Gamma(\hat{t}_i)$ for $i \in \{1, \ldots, n\}$ and use the assumption that $\{\hat{g}\}_{g \in \Delta}$ and $\{g'\}_{g \in \Delta}$ are solutions of Spec we obtain

$$f_\sigma(\hat{t}) = f_\sigma(\hat{g}(\hat{t}_1, \ldots, \hat{t}_n)) = \kappa_k(\lambda s_1. \ldots \lambda s_l. t^g_\sigma(\overline{\nu})[x_i := \underline{\hat{t}_i}][y_j := \underline{s_j}])^{\mathbb{X}, \{\hat{g}\}_{g \in \Delta}}$$
$$f_\sigma(t') = f_\sigma(g'(t_1', \ldots, t_n')) = \kappa_k(\lambda s_1. \ldots \lambda s_l. t^g_\sigma(\overline{\nu})[x_i := \underline{t_i'}][y_j := \underline{s_j}])^{\mathbb{X}, \{g'\}_{g \in \Delta}}$$

where $k = \gamma(g(\overline{\nu}))$ as specified in Spec. As σ was arbitrary this implies that $\hat{t}$ and t' have the same behaviour type. For showing (ii) and (iii) we have to distinguish cases.

Case. $f_\sigma(\hat{t})$ is a function of "observable" type $S_1 \times \cdots \times S_l \to S$ for $l \geq 0, S_1, \ldots, S_l, S \in \mathcal{S}$. Then the corresponding term on the right hand side $t^g_\sigma(\overline{\nu})$ is of observable sort $S \in \mathcal{S}$. By definition $t^g_\sigma(\overline{\nu})$ is a term in $\mathcal{E}_{\text{res}}$. Therefore $t^g_\sigma(\overline{\nu})$ cannot contain any cooperation symbol $g \in \Delta$, because $t^g_\sigma(\overline{\nu})$ is of observable sort and the operation symbols F_σ – the only operations that can transform a term of sort H into a term of observable sort – are exclusively applied to variables because $t^g_\sigma(\overline{\nu})$ is a term in $\mathcal{E}_{\text{res}}$. Furthermore any variable $x_i \in X_{\mathsf{H}}$ occurs in $t^g_\sigma(\overline{\nu})$ in a subterm of the form $F_{\sigma'}(x_i) : (\tilde{S}_1, \ldots, \tilde{S}_m, S)$ with $S \in \mathcal{S}$ and by the induction hypothesis we have $f_{\sigma'}(\hat{t}_i) = f_\sigma(t_i')$. Putting these facts together it is straightforward to prove that

$$(t^g_\sigma(\overline{\nu})[x_i := \underline{\hat{t}_i}][y_j := \underline{s_j}])^{\mathbb{X}, \{\hat{g}\}_{g \in \Delta}} = (t^g_\sigma(\overline{\nu})[x_i := \underline{t_i'}][y_j := \underline{s_j}])^{\mathbb{X}, \{g'\}_{g \in \Delta}},$$

for all $s_1 \in S_1, \ldots, s_l \in S_l$.

Case. $f_\sigma(\hat{t})$ is a function of sort $S_1 \times \cdots \times S_l \to \mathsf{H}$ for $l \geq 0, S_1, \ldots, S_l \in \mathcal{S}$. Then for $s_1 \in S_1, \ldots, s_l \in S_l$ we define a term $r \in \mathcal{T}$ by putting $r := t^g_\sigma(\overline{\nu})[x_i := t_i][y_j := \underline{s_j}]$. Spelling out the definition we obtain

$$f_\sigma(\hat{t})(s_1, \ldots, s_l) = (r)^{\mathbb{X}, \{\hat{g}\}_{g \in \Delta}} \quad \text{and} \quad f_\sigma(t')(s_1, \ldots, s_l) = (r)^{\mathbb{X}, \{g'\}_{g \in \Delta}}$$

which implies $(f_\sigma(\hat{t})(s_1, \ldots, s_l), f_\sigma(t')(s_1, \ldots, s_l)) \in R$ as required. $\square$

6.1.3. The solution

Throughout this section we fix a finite cosignature Σ, a complete, finite Σ-specification $(\mathcal{O}, E, \mathcal{B})$ of H and a well-formed system Spec of behavioural differential equations for $(\mathcal{O}, E, \mathcal{B})$.

For all $\sigma \in \Sigma$ with $a(\sigma) = \alpha_1 + \cdots + \alpha_m \in \text{Arity}(\mathcal{S})$ we define a function $F_\sigma : \mathcal{T} \to \mathcal{T}_{\alpha_1} + \cdots + \mathcal{T}_{\alpha_m}$. The F_σ's are defined by induction on the structure of the terms in $\mathcal{T}$.

Definition 53. The *term coalgebra* $\mathbb{T} = (\mathcal{T}, \langle F_\sigma : \sigma \in \Sigma \rangle)$ is defined inductively by putting for all $\sigma \in \Sigma$:

$$\begin{aligned}
F_\sigma(\underline{\tau}) \quad &:= \quad \kappa_k(\underline{f_\sigma(\tau)}) \qquad \text{with} \quad \Gamma_{\mathbb{X}}(\tau)(\sigma) = k \\
\text{where } \underline{f_\sigma(\tau)}(s_1, \ldots, s_n) \quad &:= \quad \begin{cases} \underline{f_\sigma(\tau)(s_1, \ldots, s_n)} \text{ if } f_\sigma(\tau)(s_1, \ldots, s_n) \in H \\ f_\sigma(\tau)(s_1, \ldots, s_n) \text{ otherwise} \end{cases} \\
F_\sigma(g(t_1, \ldots, t_n)) \quad &:= \quad \kappa_k(\lambda \vec{s}.(t^g_\sigma[y_j := \underline{s_j}][x_i \mapsto t_i])^{\mathbb{T}}) \\
&\qquad \text{with} \quad k = \gamma(g(\nu_1, \ldots, \nu_n)), \\
&\qquad \qquad \nu_i := \Gamma_{\mathbb{T}}(t_i) \quad \text{for} \quad i \in \{1, \ldots, n\}
\end{aligned}$$

Remark 54. In order to see that the definition of $\mathbb{T}$ is correct, one has to observe that in the inductive clause of the definition the behaviour pattern $\Gamma_{\mathbb{T}}(t_i)$ is defined because $F_\sigma(t_i)$ is defined for all $\sigma \in \Sigma$ by the inductive hypothesis.

Lemma 55. *Let $(\mathcal{O}, E, \mathcal{B})$ be a complete specification of H and let* Spec *be a well-formed system of behavioural differential equations for $(\mathcal{O}, E, \mathcal{B})$. Furthermore let $\mathbb{T} = (\mathcal{T}, \langle F_\sigma : \sigma \in \Sigma\rangle)$ be the term Σ-coalgebra defined above. For all $t \in \mathcal{T}$ and all $e \in E$ we get that $\Gamma_{\mathbb{T}}(t) \in \mathcal{B}$ and $t \models e$.*

Proof. Let t be an element of $\mathcal{T}$. The fact that $\Gamma_{\mathbb{T}}(t) \in \mathcal{B}$ can be easily proven exploiting the fact that for all $t = g(t_1, \ldots, t_n)$, $g \in \Delta$, we have $\Gamma(g(t_1, \ldots, t_n)) = \gamma(g(\nu_1, \ldots, \nu_n))$ with $\nu_i = \Gamma(t_i)$ for $i \in \{1, \ldots, n\}$. Let now $(e_1, e_2) \in E$. We have to show that

$$(e_1[x \mapsto t])^{\mathbb{T}} = (e_2[x \mapsto t])^{\mathbb{T}}. \tag{5}$$

If $x \mapsto t$ is not an admissible valuation the claim is trivially true, because both sides of the equation are equal to $\perp$. Therefore we can assume $x \mapsto t$ to be admissible. We prove that (5) holds by induction on the structure of t.

Case. $t = \underline{\tau}$ for some $\tau \in \mathbf{X}$. In order to show that (5) holds it suffices to prove that the function $(\underline{\ \ }) : H \to \mathcal{T}$ that maps an element $\tau \in H$ to the corresponding constant $\underline{\tau} \in \mathcal{T}$ is a Σ-coalgebra morphism from $\langle H, \langle f_\sigma : \sigma \in \Sigma\rangle\rangle$ to $\mathbb{T}$. This a matter of routine checking. By Lemma 45 and the fact that $\tau \models (e_1, e_2)$ it now follows that also $\underline{\tau} \models (e_1, e_2)$.

Case. $t = g(t_1, \ldots, t_n)$ for some $g \in \Delta$. Let $\nu_i := \Gamma(t_i)$ and let $x_i \in X_{\mathrm{H}}$ be a variable with behaviour pattern $\Gamma(x_i) = \nu_i$ for all $i \in \{1, \ldots, n\}$. Furthermore let α be the variable assignment that maps for all $1 \leq i \leq n$ the variable x_i to the term t_i. Then by I.H. we have $(e_1[\alpha])^{\mathbb{T}} = (e_2[\alpha])^{\mathbb{T}}$, i.e., $\mathbb{T}, \alpha \models (e_1, e_2)$, for all $e = (e_1, e_2) \in E(x_1, \ldots, x_n)$. Moreover for all $\sigma \in \Sigma$ with $a(\sigma) = \alpha_1 + \cdots + \alpha_m$ and $\alpha_{\gamma(g(\nu_1,\ldots,\nu_n))} = (S_1 \ldots S_l, T)$ and for all $(s_1, \ldots, s_l) \in S_1 \times \cdots \times S_l$ by definition we have $(F_\sigma(g(x_1, \ldots, x_n))(y_1, \ldots, y_l)[y_j := \underline{s_j}][\alpha])^{\mathbb{T}} = (t^g_\sigma(\overline{\nu})[y_j := \underline{s_j}][\alpha])^{\mathbb{T}}$ and thus $\mathbb{T}, \alpha \models (e_1, e_2)$ for all equations $e^{g'}_{\sigma'}(\overline{\nu}) = (e_1, e_2)$ in Spec. By (4) it follows that for an arbitrary $e = (e_1, e_2) \in E$ we have $\mathbb{T}, \alpha \models (e_1[x := g(x_1, \ldots, x_n)], e_1[x := g(x_1, \ldots, x_n)])$, i.e.,

$$(e_1[x := g(x_1, \ldots, x_n)][\alpha])^{\mathbb{T}} = (e_2[x := g(x_1, \ldots, x_n)][\alpha])^{\mathbb{T}},$$

which is equivalent to $(e_1[x \mapsto t])^{\mathbb{T}} = (e_2[x \mapsto t])^{\mathbb{T}}$. The latter shows that $t \models e$ as required. $\square$

The following is an immediate corollary.

Corollary 56. *There exists a unique Σ-coalgebra morphism*

$$\iota : (\mathcal{T}, \langle F_\sigma : \sigma \in \Sigma\rangle) \to (H, \langle f_\sigma : \sigma \in \Sigma\rangle).$$

Proof. The claim follows from the fact that $(H, \langle f_\sigma : \sigma \in \Sigma\rangle)$ is relatively final amongst all Σ-coalgebras that satisfy the equations in E (Lemma 49) and from the fact that the term coalgebra satisfies the equations in E (Lemma 55). $\square$

The final map ι can be used in order to obtain the solution of the given system Equ of behavioural differential equations.

Definition 57. Let $(\mathcal{O}, E, \mathcal{B})$ be a complete, finite specification of H, let Spec be a well-formed system of behavioural differential equations for $(\mathcal{O}, E, \mathcal{B})$ and let ι be the unique Σ-coalgebra morphism that exists by Corollary 56. For every $g \in \Delta$ we define a function $\hat{g} : H^n \to H$ by putting $\hat{g}(\tau_1, \ldots, \tau_n) := \iota(g(\underline{\tau}_1, \ldots, \underline{\tau}_n))$.

The above definition yields the unique solution of a given well-formed system of behavioural differential equations.

Proposition 58. *Let $(\mathcal{O}, E, \mathcal{B})$ be a complete, finite specification of H and let* Spec *be a well-formed system of behavioural differential equations for $(\mathcal{O}, E, \mathcal{B})$ and a given set of function symbols Δ. The family $\{\hat{g}\}_{g\in\Delta}$ from Definition 57 is the unique solution of* Spec.

Proof. The fact that $\{\hat{g}\}_{g\in\Delta}$ is a solution of Spec can be checked using coinduction. That the solution of a well-formed system of behavioural differential equations is unique has been proven in Proposition 52. $\square$

6.2. Definition scheme: short examples

We now give a short list of examples that are instances of our definition scheme. An example that has been worked out in more detail can be found in Section 6.3 below. The first three examples are much simpler than the formulation of the general scheme might suggest because the behaviour patterns do not play a role in case we are dealing with *basic* cosignatures.

(1) Consider the set of bi-infinite streams $\mathbb{Z}^{\mathbb{Z}}$ of integers together with the set of cooperations $\{h : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}, l : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}}, r : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}}\}$ (cf., Example 7(4)). The equations $(F_l(F_r(x)), x)$ and $(F_r(F_l(x)), x)$ can be seen to completely specify $(\mathbb{Z}^{\mathbb{Z}}, \langle h, l, r\rangle)$. The following is a well-formed system of differential equations for $\Delta = \{\sigma\} \cup \{+(\_, z) \mid z \in \mathbb{Z}\}$:

$$\begin{aligned} F_h(\sigma) &= 0 & F_h(+(x, z)) &= F_h(x) + z \\ F_l(\sigma) &= +(\sigma, 1) & F_l(+(x, z)) &= +(F_l(x), z) \\ F_r(\sigma) &= +(\sigma, -1) & F_r(+(x, z)) &= +(F_r(x), z) \end{aligned}$$

where $z \in \mathbb{Z}$. Then the functions $+(\_, z) : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}}$ for all $z \in \mathbb{Z}$ that add to a given bi-infinite stream the integer z and the constant

$$\sigma = (\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots),$$

form the unique solution.

(2) Here is an example of an $\{h, \Delta\}$-differential equation (cf., Example 7(6)):

$$\Delta\sigma = \sigma\,, \qquad \sigma(0) = 1$$

It has a unique solution:

$$\sigma = (2^0, 2^1, 2^2, \ldots).$$

A closed expression for this solution can be computed using the following identity, which can be viewed as the fundamental theorem of the difference calculus: for all $\tau \in \mathbb{R}^{\omega}$,

$$\tau = \frac{1}{1 - X} \times (\, \tau_0 + X \times \Delta\tau \,)$$

Using this and the differential equation above, one obtains

$$\sigma = \frac{1}{1 - 2X} = (2^0, 2^1, 2^2, \ldots)$$

(3) The following is an example of an $\{h, d/dX\}$-differential equation (cf., Example 7(7)):

$$\frac{d\sigma}{dX} = \sigma\,, \qquad \sigma(0) = 1$$

Again, it has a unique solution, which is now given by

$$\sigma(n) = \frac{1}{n!}$$

(It is not obvious how to find a closed expression for σ.)

(4) Coming back to Example 33 it is straightforward to prove that the given equations specify addition and multiplication on $\bar{\mathbb{N}}$. In order to fit the example into the general scheme one has to replace the two given case distinctions by 4 equations each as described in Example 37 above. Strictly speaking, it is moreover necessary to explicitly specify the behaviour patterns $\gamma(\mathrm{add}(x, y))$ and $\gamma(\mathrm{mult}(x, y))$ for all possible choices of behaviour patterns of x and y. Consider for example two variables x ad y with $\Gamma(x)(\bar{P}) = \Gamma(y)(\bar{P}) = 2$. Then we have $F_{\bar{P}}(\mathrm{add}(x, y)) = \mathrm{add}(\bar{P}(x), y)$ and $F_{\bar{P}}(\mathrm{mult}(x, y)) = \mathrm{add}(F_{\bar{P}}(x), \mathrm{mult}(F_{\bar{P}}(x), y))$. Both times on the right side of the equation is a term of sort $\mathsf{H} = \bar{\mathbb{N}}$ and therefore we put $\gamma(\mathrm{add}(x, y))(\bar{P}) = \gamma(\mathrm{mult}(x, y))(\bar{P}) = 2$ indicating that application of $F_{\bar{P}}$ yields in both cases a result in the second component of the coproduct $1 + \bar{\mathbb{N}}$.

(5) Also Example 34 fits easily into our scheme. Note that in this example we make use of the fact that we can specify a set $\mathcal{B}$ of allowed behaviour patterns (cf., Example 48). Again one has to add to the specification the behaviour patterns $\gamma(\mathrm{zip}(x, y))$ for all choices of x and y. Because of the restriction specified by $\mathcal{B}$ we only have to consider variables x that either have behaviour pattern ν_1 with $\nu_1(\mathrm{head}) = \nu_1(\mathrm{tail}) = 1$ or ν_2 with $\nu_2(\mathrm{head}) = \nu_2(\mathrm{tail}) = 2$. Let x_1, y_1 and x_2, y_2 be variables with behaviour pattern ν_1 and ν_2, respectively. Then a function γ consistent with the specification in Example 34 would be

$$\gamma(\mathrm{zip}(x_i, y_j)) := \nu_2 \qquad \text{for } i \neq 1 \text{ or } j \neq 1$$
$$\gamma(\mathrm{zip}(x_1, y_1)) := \nu_1$$

and thus $\gamma(\mathrm{zip}(x_i, y_j)) \in \mathcal{B}$ for all $i, j \in \{1, 2\}$.

6.3. *Definition scheme for* {head, even, odd}

We now want to look at one instance of our definition scheme in somewhat more detail. We consider the set $H = A^\omega$ of infinite A-streams for a given non-empty set A. For our scheme we first need a complete specification $(\mathcal{O}, E)$ of A^ω. In Section 4.3, we saw that the cooperations {head, even, odd} are complete with respect to A^ω. Thus we put $\mathcal{O} = \{\mathrm{head}, \mathrm{even}, \mathrm{odd}\}$, i.e., our cosignature consists of one constant head with $a(\mathrm{head}) = (\epsilon, A)$, and two operation symbols even and odd with $a(\mathrm{even}) = a(\mathrm{odd}) = (\epsilon, \mathsf{H})$.[3] Again we would like to remind the reader that due to the fact that we are dealing with a basic cosignature here that does not involve the coproduct there will be no need to specify behaviour patterns of states and variables.

For a complete specification of A^ω, however, we also need some equations that characterise the subcoalgebra of the final $A \times \mathrm{Id} \times \mathrm{Id}$-coalgebra that is isomorphic to $(A^\omega, \langle \mathrm{head}, \mathrm{even}, \mathrm{odd}\rangle)$: intuitively speaking this subcoalgebra consists of those binary A-labelled trees that do not change the label on paths that go to the left only – corresponding to the fact that the first element of a stream σ and the first element of $\mathrm{even}(\sigma)$ are equal. This property can be expressed by the following state equation:

$$F_{\mathrm{head}}(F_{\mathrm{even}}(x)) = F_{\mathrm{head}}(x) \text{ with some variable } x \in X_{\mathsf{H}},$$

i.e., we put $E := \{(F_{\mathrm{head}}(F_{\mathrm{even}}(x)), F_{\mathrm{head}}(x))\}$.

Recall the representation of the final $A \times \mathrm{Id} \times \mathrm{Id}$-coalgebra $(A^{2^*}, \langle h, l, r\rangle)$ from Fact 23 and let $j : A^\omega \to A^{2^*}$ be the (injective) coalgebra morphism from $(A^\omega, \langle \mathrm{head}, \mathrm{even}, \mathrm{odd}\rangle)$ into the final coalgebra.

Lemma 59. *Using the terminology of Definition 47 we have* $P_E = \{t \in A^{2^*} \mid h(t) = h(l(t))\} \subseteq A^{2^*}$ *and* $j[A^\omega] = \Box P_E$*, i.e.,* $A^\omega \cong \Box \mathbb{P}_E$*. Therefore* $(\mathcal{O}, E)$ *is a complete specification of* A^ω*.*

Proof. The first claim about P_E can be seen to be true by spelling out the definition of P_E. In order to show that $j[A^\omega] = \Box P_E$ we first prove $\Box P_E \subseteq j[A^\omega]$. Let $t \in \Box P_E$. Then it is easy to see that

$$\text{for all} \quad w \in 2^* \quad \text{we have } t_w \in P_E, \text{ i.e., } h(l(t_w)) = h(t_w). \tag{6}$$

We define a stream $\tau \in A^\omega$ by putting $\tau_n := h(t_{\mathrm{bin}(n)})$ for all $n \in \omega$. Our claim is that $j(\tau) = t$. We prove $j(\tau)(w) = t(w)$ for all $w \in 2^*$ by induction on w.

Base case. $w = \epsilon$. Then $j(\tau)(\epsilon) = \tau_0 = h(t) = t(\epsilon)$.
Case. $w = 0v$. Then

$$\begin{aligned} j(\tau)(0w) &= j(\tau)(w) \overset{\text{I.H.}}{=} t(w) = h(t_w) \\ &\overset{(6)}{=} h(l(t_w)) = h(t_{0w}) = t(0w) \end{aligned}$$

Case. $w = 1v$. Then

$$j(\tau)(1w) = \tau_{\mathrm{nat}(1w)} \overset{\text{Def.}}{=} h(t_{\mathrm{bin}(\mathrm{nat}(1w))}) = h(t_{1w}) = t(1w)$$

This concludes the proof of $\Box P_E \subseteq j[A^\omega]$. For the converse direction note that obviously $j[A^\omega] \subseteq P_E$. Because j is a homomorphism and hence $j[A^\omega]$ is a subcoalgebra of $(A^{2^*}, \langle h, l, r\rangle)$ we get $j[A^\omega] \subseteq \Box P_E$ as required. $\Box$

Now we are ready to concretely describe the stream definition scheme. Given a set of functions symbols Δ, each $g \in \Delta$ with an arity $a(g) \in \mathbb{N}$, the syntax for the definition scheme is defined as above - but now for the special case that $H = A^\omega$, $\Sigma = \{\mathrm{head}, \mathrm{even}, \mathrm{odd}\}$ and $\mathcal{S} = \{A\}$. Then a *well-formed* system of behavioural differential equations for $(\mathcal{O}, E)$ and Δ is a set Equ of equations which contains for every $g \in \Delta_n$ three equations

[3] Note that we simply write head, even and odd instead of f_{head}, f_{even} and f_{odd}.

$$\begin{aligned} F_{\text{head}}(g(x_1,\ldots,x_n)) &:= c^g(F_{\text{head}}(x_1),\ldots,F_{\text{head}}(x_n)) \\ &\qquad \text{for some function } c : A^n \to A \\ F_{\text{even}}(g(x_1,\ldots,x_n)) &:= t^g_{\text{even}}(x_1,\ldots,x_n) \\ F_{\text{odd}}(g(x_1,\ldots,x_n)) &:= t^g_{\text{odd}}(x_1,\ldots,x_n) \end{aligned}$$

where t^g_{even} and t^f_{odd} are terms in $\mathcal{E}_{\text{res}}$ with variables contained in $\{x_1,\ldots,x_n\}$. Furthermore we require that we can prove for all $g \in \Delta$ the following conditional equation

$$F_{\text{head}}(F_{\text{even}}(g(\vec{x}))) = F_{\text{head}}(g(\vec{x})) \quad \Leftarrow \quad \mathsf{Equ} \cup \{F_{\text{head}}(F_{\text{even}}(x_i)) = F_{\text{head}}(x_i) \mid x_i \in X\}.$$

By Corollary 56 there exists a unique coalgebra morphism

$$\iota : (\mathcal{T}, \langle F_{\text{head}}, F_{\text{even}}, F_{\text{odd}}\rangle) \to (A^\omega, \langle \text{head}, \text{even}, \text{odd}\rangle),$$

i.e., ι makes the following diagram commute:

$$\begin{tikzcd} \mathcal{T} \arrow[r, dashed, "\iota"] \arrow[d, "\langle F_{\text{head}}, F_{\text{even}}, F_{\text{odd}}\rangle"'] & A^\omega \arrow[d, "\langle \text{head}, \text{even}, \text{odd}\rangle"] \\ A \times \mathcal{T} \times \mathcal{T} \arrow[r, dashed, "\text{id}\times\iota\times\iota"'] & A \times A^\omega \times A^\omega \end{tikzcd}$$

Furthermore the function ι can be used in order to compute the unique *solution* for the given set Equ of behavioural differential equations:

Proposition 60. *Let* Equ *be a well-formed system of behavioural differential equations for a given set* Δ *of function symbols and let* $\iota : \mathcal{T} \to A^\omega$ *be the coalgebra map that interprets terms* $t \in \mathcal{T}$ *as A-streams. Furthermore we define for every* $a(g)$*-ary function symbol* $g \in \Delta$ *a function* $\hat{g} : (A^\omega)^{a(g)} \to A^\omega$ *by putting* $\hat{g}(\tau_1,\ldots,\tau_{a(g)}) := \iota(g(\underline{\tau_1},\ldots,\underline{\tau_{a(g)}}))$*. Then the family* $\{\hat{g}\}_{g\in\Delta}$ *is the (unique) solution of* Equ.

Proof. This is just a special case of Proposition 58 above. □

As an example recall the definition of the Thue–Morse sequence from Example 32.

Example 61. Let $A = 2$ and $\Delta = \{\text{inv}, \text{TM}\}$. We define Equ to consist of the following set of equations

$$\begin{aligned} F_{\text{head}}(\text{inv}(x)) &:= 1 - F_{\text{head}}(x) & \qquad F_{\text{head}}(\text{TM}) &:= 0 \\ F_{\text{even}}(\text{inv}(x)) &:= \text{inv}(F_{\text{even}}(x)) & F_{\text{even}}(\text{TM}) &:= \text{TM} \\ F_{\text{odd}}(\text{inv}(x)) &:= \text{inv}(F_{\text{odd}}(x)) & F_{\text{odd}}(\text{TM}) &:= \text{inv}(\text{TM}) \end{aligned}$$

In order to see that this system of equations is well formed one can easily check that the following conditional equations are theorems of conditional equational logic

$$\begin{aligned} F_{\text{head}}(F_{\text{even}}(TM)) = F_{\text{head}}(TM) &\quad\Leftarrow\quad F_{\text{even}}(TM) = TM \\ F_{\text{head}}(F_{\text{even}}(\text{inv}(x))) = F_{\text{head}}(\text{inv}(x)) &\quad\Leftarrow\quad \{F_{\text{head}}(F_{\text{even}}(x)) = F_{\text{head}}(x)\} \\ &\qquad\quad \cup\ \mathsf{Equ} \end{aligned}$$

The unique solution of this system of equations consists of the function $\text{inv} : 2^\omega \to 2^\omega$ that inverts a given bitstream and of the constant $\text{TM} : 1 \to 2^\omega$ which is the so-called Thue–Morse sequence.

We close this section with some more examples for defining streams and stream functions using {head, even, odd}.

Example 62. This example demonstrates that we can define Stern's diatomic series (see, e.g., [6], pp. 230–232, where this sequence is called fusc) using {head, even, odd} as a complete set of cooperations. Let $A = 2$ and $\Delta = \{\text{Stern}, \text{tern}, \text{add}\}$

and consider the following system of equations:

$$\begin{aligned} F_{\mathrm{head}}(\mathrm{Stern}) &:= 0 & F_{\mathrm{head}}(\mathrm{tern}) &:= 1 \\ F_{\mathrm{even}}(\mathrm{Stern}) &:= \mathrm{Stern} & F_{\mathrm{even}}(\mathrm{tern}) &:= \mathrm{add}(\mathrm{Stern}, \mathrm{tern}) \\ F_{\mathrm{odd}}(\mathrm{Stern}) &:= \mathrm{add}(\mathrm{Stern}, \mathrm{tern}) & F_{\mathrm{odd}}(\mathrm{tern}) &:= \mathrm{tern} \end{aligned}$$

$$\begin{aligned} F_{\mathrm{head}}(\mathrm{add}(x, y)) &:= F_{\mathrm{head}}(x) + F_{\mathrm{head}}(y) \\ F_{\mathrm{even}}(\mathrm{add}(x, y)) &:= \mathrm{add}(F_{\mathrm{even}}(x), F_{\mathrm{even}}(y)) \\ F_{\mathrm{odd}}(\mathrm{add}(x, y)) &:= \mathrm{add}(F_{\mathrm{odd}}(x), F_{\mathrm{odd}}(y)) \end{aligned}$$

Then Stern equal to Stern's diatomic series, tern equal to the tail of Stern and add equal to the function adding two streams will be the solution of this system of equation.

Example 63. More definitions for $A = \mathbb{R}$ using {head, even, odd}:

$$\begin{aligned} F_{\mathrm{head}}(\mathrm{zip}(x, y)) &:= F_{\mathrm{head}}(x) & F_{\mathrm{head}}(\sigma(X^2)) &:= F_{\mathrm{head}}(\sigma) \\ F_{\mathrm{even}}(\mathrm{zip}(x, y)) &:= x & F_{\mathrm{even}}(\sigma(X^2)) &:= \sigma \\ F_{\mathrm{odd}}(\mathrm{zip}(x, y)) &:= y & F_{\mathrm{odd}}(\sigma(X^2)) &:= 0 \end{aligned}$$

$$\begin{aligned} F_{\mathrm{head}}(\sigma(-X)) &:= F_{\mathrm{head}}(\sigma) & F_{\mathrm{head}}(X \times \sigma) &:= 0 \\ F_{\mathrm{even}}(\sigma(-X)) &:= F_{\mathrm{even}}(\sigma) & F_{\mathrm{even}}(X \times \sigma) &:= X \times F_{\mathrm{odd}}(\sigma) \\ F_{\mathrm{odd}}(\sigma(-X)) &:= -F_{\mathrm{odd}}(\sigma) & F_{\mathrm{odd}}(X \times \sigma) &:= F_{\mathrm{even}}(\sigma) \end{aligned}$$

Furthermore it easy to define componentwise addition (+), multiplication ($\times$) and subtraction ($-$). Note that we write $\sigma(X^2)$, $\sigma(-X)$ and $X \times \sigma$ in order to stay consistent with the commonly used notation from stream calculus. In these cases σ is the variable and we define functions g_1, g_2 and g_3 with $g_1(\sigma) = \sigma(X^2)$, $g_2(\sigma) = \sigma(-X)$ and $g_3(\sigma) = X \times \sigma$.

The last example also provides us with a further illustration of {head, even, odd}-coinduction.

Example 64. Given the definitions in Example 63 we want to prove that

$$\frac{1}{2}(\sigma + \sigma(-X)) = \mathrm{even}(\sigma)(X^2). \tag{7}$$

By Proposition 29 and the fact that {head, even, odd} is a complete set of cooperations it suffices to show that $\mathrm{head}(\sigma) = \mathrm{head}(\tau)$, $\mathrm{even}(\sigma) = \mathrm{even}(\tau)$ and $\mathrm{odd}(\sigma) = \mathrm{odd}(\tau)$ in order to prove that $\sigma = \tau$ for streams σ, τ. We compute

$$\begin{aligned} \mathrm{head}\left(\frac{1}{2}(\sigma + \sigma(-X))\right) &= \frac{1}{2}\,(\mathrm{head}(\sigma) + \mathrm{head}(\sigma(-X))) \\ &= \frac{1}{2}\,(\mathrm{head}(\sigma) + \mathrm{head}(\sigma)) = \mathrm{head}(\sigma) \\ &= \mathrm{head}(\mathrm{even}(\sigma)(X^2)) \\ \mathrm{even}\left(\frac{1}{2}(\sigma + \sigma(-X))\right) &= \frac{1}{2}\,(\mathrm{even}(\sigma) + \mathrm{even}(\sigma(-X))) \\ &= \frac{1}{2}(\mathrm{even}(\sigma) + \mathrm{even}(\sigma)) = \mathrm{even}(\sigma) \\ &= \mathrm{even}(\mathrm{even}(\sigma)(X^2)) \\ \mathrm{odd}\left(\frac{1}{2}(\sigma + \sigma(-X))\right) &= \frac{1}{2}\,(\mathrm{odd}(\sigma) + \mathrm{odd}(\sigma(-X))) = \frac{1}{2}\,(\mathrm{odd}(\sigma) - \mathrm{odd}(\sigma)) \\ &= 0 = \mathrm{odd}(\mathrm{even}(\sigma)(X^2)) \end{aligned}$$

Therefore we can conclude by {head, even, odd}-coinduction that (7) holds. We also can use {head, even, odd}-coinduction in order to prove the following:

$$\mathrm{zip}(\sigma, \tau) = \sigma(X^2) + X \times \tau(X^2). \tag{8}$$

Again we show that both sides behave the same if we apply head, even and odd.

$$\begin{aligned}
\mathrm{head}(\sigma(X^2)+X\times\tau(X^2)) &= \mathrm{head}(\sigma)=\mathrm{head}(\sigma(X^2))+0=\mathrm{head}(\mathrm{zip}(\sigma,\tau))\\
\mathrm{even}(\sigma(X^2)+X\times\tau(X^2)) &= \mathrm{even}(\sigma(X^2))+\mathrm{even}(X\times\tau(X^2))\\
&= \sigma+X\times\mathrm{odd}(\tau(X^2))\\
&\overset{X\times 0=0}{=} \sigma=\mathrm{even}(\mathrm{zip}(\sigma,\tau))\\
\mathrm{odd}(\sigma(X^2)+X\times\tau(X^2)) &= \mathrm{odd}(\sigma(X^2)+\mathrm{odd}(X\times\tau(X^2))\\
&= 0+\mathrm{even}(\tau(X^2))=\tau=\mathrm{odd}(\mathrm{zip}(\sigma,\tau))
\end{aligned}$$

7. Related and future work

7.1. Connection with Hidden algebra

An important source of inspiration for this paper was the work on hidden algebra (cf., e.g., [15]) and its close connection to coalgebra which has been described in the papers by Cîrstea (cf., [4,5]). A hidden specification consists of a many-sorted algebraic signature Σ, involving *observable* and *hidden* sorts, together with a set of equations that specify certain constraints on the given operations. The notion of a *cobasis* from hidden algebra defines when a given set of operations is "complete". If we think of the operations as ways for obtaining information about elements of hidden sort, completeness means that we can either distinguish two given elements of some hidden sort using the operations of the cobasis, or these elements should be considered to be equal.

Example 65 (*Sketch*)**.** A possible hidden specification for streams over a set A contains the operations {head, cons, even, odd, tail, zip} together with the equations that are to be expected (cf., e.g., [14]). The sorts in this example are Stream and A where Stream is the hidden sort. Possible cobases would be: the set of all operations, the set {head, tail} and the set {head, even, odd}. But for example {head, even} would not be a cobasis.

It follows from the results in [4] that the Σ-coalgebras for a basic cosignature Σ can be seen as hidden algebras. Cobases are closely related to complete sets of cooperations, but these two notions do not coincide: a cobasis is defined for a given specification and hence for *all* hidden algebras (or Σ-coalgebras) that are a model for this specification. In the above example the set of A-streams can be seen as one model of the specification. Thus any cobasis for the specification will give rise to a complete set of cooperations on the set A^ω. Complete sets of cooperations are defined relative to one given set only. The difference between cobases and complete set of cooperations is demonstrated by the following example.

Example 66 (*Sketch continued*)**.** In hidden algebra, streams over A can be also specified using the operations head, tail and cons only. Obviously the set {head, even, odd} is a complete set of cooperations for A^ω but not a cobasis for the {head, tail, cons}-specification of streams. The reason for the latter fact is that it is not difficult to see that not every hidden {head, tail, cons}-algebra can be extended to a {head, tail, cons, even, odd}-algebra.

Summarising one could say that our definition of a complete set of cooperations is more basic then the notion of a cobasis. Nevertheless it gives rise to interesting coinductive definition and proof principles as we hope to have demonstrated.

7.2. Future work

We believe that the value of our definition scheme lies in the fact that it is parametric in the type of objects under consideration and in the (complete) given set of cooperations. The generality of our approach, however, has the drawback that for concrete cases, approaches which have been designed explicitly for these cases put less restrictions on the format of a "correct" definition. We are thinking, for example, of the recent work on defining streams and stream functions in [7] where techniques from (infinite) term rewriting are employed. At the moment we are working on making our definition scheme more liberal, mainly by using a refined induction argument for defining the term coalgebra (cf., Definition 53). Furthermore we want to formally explore possible differences between different sets of cooperations on a given set of objects. One question is, for example, whether one complete set of cooperations allows to define more or different functions on streams than another one.

Acknowledgments

The authors thank Alexandra Silva for valuable suggestions and discussions. Furthermore we are grateful to the anonymous referees for providing a number of very helpful comments.

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